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Galilean covariant theories for Bargmann–Wigner fields with arbitrary spin

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Abstract
We construct Lagrangians for non-relativistic massive fields with arbitrary spin. We use a Bargmann–Wigner construction, together with a Galilean covariant approach based on the reduction from an extended (4, 1) Minkowski manifold to the Galilean (3, 1) spacetime. Fierz identities are developed within this framework. By using symmetric spinor fields of rank 2 and rank 3, we can avoid the difficulty arising from the introduction of the minimal electromagnetic interaction in the Bargmann–Wigner wave equations. For fields with spin S, the minimal electromagnetic coupling thereby leads to the gyromagnetic ratio $g_S = 1/S$.

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1. Introduction

In non-relativistic quantum mechanics (NQM), Hagen and Hurley proposed the Galilean-covariant Bargmann–Wigner (BW) theory in terms of the multi-spinor field of rank $2S$, where S is the spin of the field [1]. The $1/S$ conjecture of the gyromagnetic ratio by Belinfante [2] was also obtained by introducing the minimal electromagnetic interaction into the Galilean-covariant BW wave equation. It is well known that in relativistic quantum mechanics (RQM), a set of the BW equations becomes inconsistent when the minimal electromagnetic coupling is introduced [3]. This inconsistency occurs even in NQM for the BW wave equations coupled with the minimal electromagnetic interaction. As strongly suggested in [3], a way to avoid this difficulty is to construct a Lagrangian formulation for non-relativistic fields in terms of the symmetric spinor fields of rank 2 or rank 3.

The purpose of this paper is to construct Lagrangians for BW fields with arbitrary spin in the Galilean-covariant manner. The method, which is applied hereafter to a massive field theory, entails such parallel results as the development of Fierz identities in a Galilean framework. A reason to persist in using the Lagrangian formalism is that invariance of the equation of motion does not always imply the conservation law if the equation of motion is not derivable from a Lagrangian (see, for instance, the introduction of reference [3]). To this end, we employ a covariant form of Galilei transformations of five coordinates $\mathbf{x}$, t and an extra variable s [4, 5]. Mathematically, this additional parameter corresponds to a central extension of the Galilei group, and thereby it enables us to deal only with vector representations of the Galilei group, rather than its projective representations. The main advantage gained here is that we can proceed in a way quite analogous to RQM. Indeed, many of our non-relativistic equations have the same form as the corresponding equations in RQM except that they are written in a manifestly covariant form on a Minkowski space in (4, 1) dimensions. This is motivated by the fact that the Galilei group in (3, 1) spacetime is a subgroup of the Poincaré group in (4, 1) spacetime.

Following reference [4, 5], let us define *Galilean 5-vectors* $(\mathbf{x}, x^4, x^5)$ as transforming under Galilean boosts as

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} - \mathbf{V}x^4, \\ x^{4'} &= x^4, \\ x^{5'} &= x^5 - \mathbf{V}\cdot\mathbf{x} + \tfrac{1}{2}\mathbf{V}^2 x^4, \end{aligned} \tag{1}$$

with the relative velocity $\mathbf{V}$. This transformation leaves the scalar product $A \cdot B = \mathbf{A}\cdot\mathbf{B} - A_4B_5 - A_5B_4$ invariant, so that, instead of constructing field models based on Lorentz covariance, we replace the Lorentz metric with the *Galilean metric*:

$$\eta_{\mu\nu} = \begin{pmatrix} \mathbf{1}_{3\times 3} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

This metric may be diagonalized to the Lorentz metric of (4, 1) spacetime; hence the unification of this formalism, which encompasses both Galilean and relativistic kinematics in (3, 1) dimensions, as emphasized in [4]. Once a covariant Lagrangian has been constructed, we must use an appropriate projection onto the (3, 1) Galilean spacetime:

$$x^\mu = (\mathbf{x}, t, s) \to (\mathbf{x}, t). \tag{2}$$

A feature of the formalism is that the extra degrees of freedom introduced through the variable s should be eliminated by means of suitable boundary conditions.

The canonical conjugate variables of the extended coordinates provide a transparent interpretation of the additional parameter s. Indeed, the 5-momentum

$$p_\mu = -\mathrm{i}\partial_\mu = (-\mathrm{i}\nabla, -\mathrm{i}\partial_t, -\mathrm{i}\partial_s) = (\mathbf{p}, -E, -m), \tag{3}$$

such that $p^4 = -p_5 = m$ and $p^5 = -p_4 = E$, shows that the coordinate s is conjugate to the mass m in the same way that $\mathbf{x}$ is conjugate to the momentum. When it comes to projecting the fields, we find, for instance with scalar fields, that the relation $\partial_s = -\mathrm{i}m$ implies the ansatz

$$\Phi(x) \equiv \mathrm{e}^{-\mathrm{i}ms}\varphi(\mathbf{x}, t). \tag{4}$$

The invariant $p^\mu p_\mu = -\kappa_m^2$, where κ_m is some constant, leads to the dispersion relation

$$E = \frac{1}{2m}\mathbf{p}\cdot\mathbf{p} + \frac{1}{2m}\kappa_m^2.$$

This takes on the familiar form, $E = \frac{1}{2m}\mathbf{p}\cdot\mathbf{p} + mc^2$, if $\kappa_m^2 = 2m^2c^2$, or $\kappa_m = \pm\sqrt{2}mc$, where m is the mass and c has units of velocity. Hereafter, we will take $c = 1$, so that

$$\kappa_m = \sqrt{2}m. \tag{5}$$

Other papers using this five-dimensional approach are in [6].

This paper is organized as follows: Kamefuchi and Takahashi have constructed the Lagrangians for the relativistic BW fields with spin 1 and spin $3/2$ in terms of the symmetric spinors of rank 2 and rank 3, respectively. We reformulate the BW constructions in the extended (4, 1) manifold for symmetric spinor fields of rank 2 in section 2, and for rank 3 in section 3. Then we construct a Lagrange formulation for non-relativistic BW fields with arbitrary spin by using the symmetric spinor fields of rank 2 and rank 3 in section 4.

We briefly discuss an inconsistency arising from the introduction of the minimal electromagnetic coupling into the BW wave equations in section 5. The Schrödinger equation follows from a reduction of the Klein–Gordon equation in the extended space. The gyromagnetic ratio obtained in section 6 is thus identical to the one found in the literature [1, 7, 8]. The final section contains a short discussion. In appendix A, we establish the Fierz identities together with useful relations for the Galilean gamma matrices, which play a crucial role in our calculations. We give the expressions of Lagrangians and the corresponding Klein–Gordon divisors without using the charge conjugation operator in appendix B.

2. Symmetric spinor field of rank 2

Our purpose is to construct a Lagrangian which leads to the equation of motion:

$$(\gamma \cdot \partial + \kappa_m)_\alpha{}^{\alpha_1} \Phi_{\alpha_1\alpha_2}(x) = 0$$

together with the symmetry property of the BW field:

$$\Phi_{\alpha_1\alpha_2}(x) = \Phi_{\alpha_2\alpha_1}(x).$$

In the first expression the dot product represents a sum over the indices $1, \ldots, 5$. Henceforth, we adhere to the notation of having lower case indices from the beginning of the Greek alphabet, α, β, γ, etc denote spinor indices and run from 1 to 4, whereas lower case indices from the middle of the Greek alphabet, κ, λ, μ, etc denote Galilean tensor indices and run from 1 to 5. We choose $c = 1$ and $\hbar = 1$.

Following reference [4], we take the four-dimensional gamma matrices as

$$\gamma = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \tag{6}$$

where the 2×2 Pauli matrices are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

The matrices of equation (6) obey the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{8}$$

where $\eta^{\mu\nu}$ is the Galilean metric. For future reference, let us define an analogue to the usual matrix γ^0:

$$\xi = \frac{-\mathrm{i}}{\sqrt{2}}(\gamma^4 + \gamma^5) = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{9}$$

In order to quantize the BW field, we adopt the Takahashi–Umezawa formulation [9]. Consider the Lagrangian

$$L(x) = -\overline{\Phi}^{\alpha_2\alpha_1}(x)\Lambda_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}(\partial)\Phi_{\beta_1\beta_2}(x), \tag{10}$$

where

$$\sqrt{2}\Lambda_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}(\partial) = \tfrac{1}{4}\mathrm{i}[(\sigma^{\mu\nu}C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\nu)^{\beta_2\beta_1} - (\gamma_\nu C)_{\alpha_1\alpha_2}(C^{-1}\sigma^{\mu\nu})^{\beta_2\beta_1}]\partial_\mu \\ + \kappa_m\tfrac{1}{4}\big[(C)_{\alpha_1\alpha_2}(C^{-1})^{\beta_2\beta_1} + (\gamma^\mu C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\mu)^{\beta_2\beta_1} + \tfrac{1}{2}(\sigma^{\mu\nu}C)_{\alpha_1\alpha_2}(C^{-1}\sigma_{\mu\nu})^{\beta_2\beta_1}\big]$$

and the adjoint field of $\Phi_{\alpha_1\alpha_2}$, denoted by $\overline{\Phi}^{\alpha_2\alpha_1}$, is given by

$$\overline{\Phi}^{\alpha_2\alpha_1}(x) = \Phi^{\dagger\alpha_2'\alpha_1'}(x)\xi_{\alpha_1'}{}^{\alpha_1}\xi_{\alpha_2'}{}^{\alpha_2}. \tag{11}$$

The matrices $(\gamma^\mu C)_{\alpha_1\alpha_2}$ and $(\sigma^{\mu\nu}C)_{\alpha_1\alpha_2}$ are symmetric with respect to the spinor indices α_1 and α_2. This can be proved thanks to the properties of the charge conjugation operator C:

$$C = -C^T$$

and

$$C^{-1} = C^\dagger$$

(see appendix A, equation (A.1)).

We may expand $\Phi_{\alpha_1\alpha_2}$ as

$$\Phi_{\alpha_1\alpha_2}(x) = \sqrt{\frac{m}{2}}\left[(C)_{\alpha_1\alpha_2}\varphi(x) + (\gamma^\mu C)_{\alpha_1\alpha_2}\varphi_\mu(x) + \frac{1}{2}(\sigma^{\mu\nu}C)_{\alpha_1\alpha_2}\varphi_{\mu\nu}(x)\right], \tag{12}$$

with

$$\varphi(x) = \sqrt{\frac{2}{m}}\frac{1}{4}(C^{-1})^{\alpha_2\alpha_1}\Phi_{\alpha_1\alpha_2}(x),$$

$$\varphi_\mu(x) = \sqrt{\frac{2}{m}}\frac{1}{4}(C^{-1}\gamma_\mu)^{\alpha_2\alpha_1}\Phi_{\alpha_1\alpha_2}(x),$$

and

$$\varphi_{\mu\nu}(x) = \sqrt{\frac{2}{m}}\frac{1}{4}(C^{-1}\sigma_{\mu\nu})^{\alpha_2\alpha_1}\Phi_{\alpha_1\alpha_2}(x),$$

where φ, φ_μ and $\varphi_{\mu\nu}$ contain operators. If we substitute the Hermitian conjugate of $\Phi_{\alpha_1\alpha_2}$ into equation (11), we find

$$\overline{\Phi}^{\alpha_2\alpha_1}(x) = \sqrt{\frac{m}{2}}\left[-\overline{\varphi}(x)(C^{-1})^{\alpha_2\alpha_1} + \overline{\varphi}^\mu(x)(C^{-1}\gamma_\mu)^{\alpha_2\alpha_1} - \overline{\varphi}^{\mu\nu}(x)\frac{1}{2}(C^{-1}\sigma_{\mu\nu})^{\alpha_2\alpha_1}\right],$$

where

$$\overline{\varphi}(x) = \varphi^\dagger(x), \qquad \overline{\varphi}^\mu(x) = \eta^{\mu\nu}\varphi_\nu^\dagger(x),$$

and

$$\overline{\varphi}^{\mu\nu}(x) = \eta^{\mu\rho}\eta^{\nu\sigma}\varphi_{\rho\sigma}^\dagger(x).$$

Let us note the following relation:

$$C\xi^T C^{-1} = -\xi.$$

The Euler–Lagrange equation of motion follows from equation (10):

$$\Lambda_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}(\partial)\Phi_{\beta_1\beta_2}(x) = 0. \tag{13}$$

If we substitute equation (12) into the equation of motion (13), and multiply by $\frac{1}{4}(C^{-1})^{\alpha_2\alpha_1}$, $\frac{1}{4}(C^{-1}\gamma_\mu)^{\alpha_2\alpha_1}$ and $\frac{1}{4}(C^{-1}\sigma_{\mu\nu})^{\alpha_2\alpha_1}$, we obtain, respectively,

$$\varphi(x) = 0,$$

$$-\mathrm{i}\partial_\nu\varphi^\nu{}_\mu(x) + \kappa_m\varphi_\mu(x) = 0,$$

$$\text{and} \quad \mathrm{i}[\partial_\mu\varphi_\nu(x) - \partial_\nu\varphi_\mu(x)] + \kappa_m\varphi_{\mu\nu}(x) = 0.$$

Therefore, we have

$$\Phi_{\alpha_1\alpha_2}(x) = -\sqrt{\frac{m}{2}}\frac{1}{\kappa_m}[(\gamma\cdot\partial - \kappa_m)\gamma^\mu C]_{\alpha_1\alpha_2}\varphi_\mu(x),$$

with

$$\left(\partial^2\delta_\mu{}^\nu - \partial_\mu\partial^\nu - \kappa_m^2\delta_\mu{}^\nu\right)\varphi_\nu(x) = 0.$$

Then $\Phi_{\alpha_1\alpha_2}$ is clearly symmetric, and satisfies

$$(\gamma\cdot\partial + \kappa_m)_\alpha{}^{\alpha_1}\Phi_{\alpha_1\alpha_2}(x) = 0.$$

Now, let us cast the equation of motion into the form (13) and seek the reciprocal operator of $\Lambda(\partial)$ defined by

$$\Lambda_{\alpha_1\alpha_2}{}^{\gamma_1\gamma_2}(\partial)d_{\gamma_1\gamma_2}{}^{\beta_1\beta_2}(\partial) = \left(\partial^2 - \kappa_m^2\right)I_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}.$$

The operator $d(\partial)$ is called 'Klein–Gordon divisor'. Let us rewrite $\Lambda(\partial)$ as

$$\Lambda_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}(\partial) = \frac{1}{\sqrt{2}}\left[(\Gamma^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\partial_\mu + \kappa_m I_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right],$$

where

$$I_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = \frac{1}{4}\left[(C)_{\alpha_1\alpha_2}(C^{-1})^{\beta_2\beta_1} + (\gamma^\mu C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\mu)^{\beta_2\beta_1} + \frac{1}{2}(\sigma^{\mu\nu}C)_{\alpha_1\alpha_2}(C^{-1}\sigma_{\mu\nu})^{\beta_2\beta_1}\right],$$

and $\quad (\Gamma^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = \frac{\mathrm{i}}{4}[(\sigma^{\mu\nu}C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\nu)^{\beta_2\beta_1} - (\gamma_\nu C)_{\alpha_1\alpha_2}(C^{-1}\sigma^{\mu\nu})^{\beta_2\beta_1}].$

It is worth noting that $\Lambda(\partial)$ can be written into a more familiar form by utilizing the Fierz identities, equations (A.2) and (A.8), in appendix A:

$$I_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = \delta_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2}, \tag{14}$$

and $\quad (\Gamma^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = \frac{1}{4}\left[(\gamma^\mu)_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2} + (\gamma^\mu)_{\alpha_1}{}^{\beta_2}\delta_{\alpha_2}{}^{\beta_1} + \delta_{\alpha_1}{}^{\beta_2}(\gamma^\mu)_{\alpha_2}{}^{\beta_1} + \delta_{\alpha_1}{}^{\beta_1}(\gamma^\mu)_{\alpha_2}{}^{\beta_2}\right]. \quad (15)$

Then we can express the Klein–Gordon divisor as

$$\begin{aligned} d_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}(\partial) &= \sqrt{2}\left[\frac{1}{\kappa_m}\left(\partial^2 - \kappa_m^2\right)I_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\Gamma^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\partial_\mu - \frac{1}{\kappa_m}(\Gamma^\mu)_{\alpha_1\alpha_2}{}^{\gamma_1\gamma_2}\partial_\mu(\Gamma^\nu)_{\gamma_1\gamma_2}{}^{\beta_1\beta_2}\partial_\nu\right], \\ &= -\frac{\sqrt{2}}{\kappa_m}\frac{1}{4}\Big[(\gamma\cdot\partial - \kappa_m)_{\alpha_1}{}^{\beta_1}(\gamma\cdot\partial - \kappa_m)_{\alpha_2}{}^{\beta_2} + (\gamma\cdot\partial - \kappa_m)_{\alpha_1}{}^{\beta_2}(\gamma\cdot\partial - \kappa_m)_{\alpha_2}{}^{\beta_1} \\ &\quad + \left(\partial^2 - \kappa_m^2\right)\left(-3\delta_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2} + \delta_{\alpha_1}{}^{\beta_2}\delta_{\alpha_2}{}^{\beta_1}\right)\Big], \end{aligned}$$

by using equations (14) and (15).

Now we quantize the BW field Φ by using the Klein–Gordon divisor, following the development by Takahashi and Umezawa [9]:

$$\begin{aligned} &[\Phi_{\alpha_1\alpha_2}(x), \overline{\Phi}^{\beta_2\beta_1}(y)] = -\mathrm{i}d_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}(\partial)\Delta(x-y,\kappa_m), \\ &\text{and} \quad \langle 0|T(\Phi_{\alpha_1\alpha_2}(x)\overline{\Phi}^{\beta_2\beta_1}(y))|0\rangle = -\mathrm{i}d_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}(\partial)\Delta_C(x-y,\kappa_m), \end{aligned} \tag{16}$$

with

$$\Delta(x-y,\kappa_m) = -\frac{\mathrm{i}}{(2\pi)^3}\int_{-\infty}^{\infty}\mathrm{d}^5p\,\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\delta\left(p^2 + \kappa_m^2\right)2m\delta((p_5)^2 - m^2), \tag{17}$$

and

$$\Delta_C(x-y,\kappa_m) = -\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\mathrm{d}^5p\,\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\frac{1}{p^2 + \kappa_m^2 - \mathrm{i}\epsilon}2p_5\delta((p_5)^2 - m^2), \tag{18}$$

where the limit $\epsilon \to 0$ is understood. Note that we find

$$[\varphi_\mu(x), \overline{\varphi}_\nu(y)] = \mathrm{i}\left(\eta_{\mu\nu} - \frac{1}{\kappa_m^2}\partial_\mu\partial_\nu\right)\Delta(x-y,\kappa_m)$$

by multiplying equation (16) by $\frac{1}{4}(C^{-1}\gamma_\mu)^{\alpha_2\alpha_1}$ and $\frac{1}{4}(\gamma_\nu C)_{\beta_1\beta_2}$ from both sides.

3. Symmetric spinor field of rank 3

The algorithm developed in section 2 can be extended to the totally symmetric spinor field of rank 3, which is assumed to satisfy

$$(\gamma\cdot\partial+\kappa_m)_\alpha{}^{\alpha_1}\Psi_{\alpha_1\alpha_2\alpha_3}(x)=0,$$

where the spinor indices satisfy the following symmetry condition:

$$\Psi_{\alpha_1\alpha_2\alpha_3}(x)=\Psi_{\alpha_2\alpha_1\alpha_3}(x)=\Psi_{\alpha_1\alpha_3\alpha_2}(x).$$

These equations can be obtained from a Lagrangian given by

$$L(x)=-\overline{\Psi}^{\alpha_3\alpha_2\alpha_1}(x)\Lambda_{\alpha_1\alpha_2\alpha_3}{}^{\beta_1\beta_2\beta_3}(\partial)\Psi_{\beta_1\beta_2\beta_3}(x), \tag{19}$$

where

$$\begin{aligned}\Lambda_{\alpha_1\alpha_2\alpha_3}{}^{\beta_1\beta_2\beta_3}(\partial)=&\bigg\{(\Gamma^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\partial_\mu+\kappa_m I_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\\&+\frac{1}{4}(\gamma^\mu C)_{\alpha_1\alpha_2}\frac{1}{\kappa_m}\big[\big(\partial^2-\kappa_m^2\big)\eta_{\mu\nu}-\partial_\mu\partial_\nu\big](C^{-1}\gamma^\nu)^{\beta_2\beta_1}\bigg\}\delta_{\alpha_3}{}^{\beta_3}\\&+\frac{1}{4}(\gamma^\mu C)_{\alpha_1\alpha_2}[\Lambda_{\mu\nu}(\partial)]_{\alpha_3}{}^{\beta_3}(C^{-1}\gamma^\nu)^{\beta_2\beta_1},\end{aligned} \tag{20}$$

with

$$\Lambda_{\mu\nu}(\partial)=(\gamma\cdot\partial+\kappa_m)\big(\eta_{\mu\nu}-\tfrac{1}{4}\gamma_\mu\gamma_\nu\big)-\tfrac{1}{4}(\gamma_\mu\partial_\nu-\gamma_\nu\partial_\mu),$$

and the adjoint field

$$\overline{\Psi}^{\alpha_3\alpha_2\alpha_1}(x)=\Psi^{\dagger\alpha_3'\alpha_2'\alpha_1'}(x)\xi_{\alpha_1'}{}^{\alpha_1}\xi_{\alpha_2'}{}^{\alpha_2}\xi_{\alpha_3'}{}^{\alpha_3}.$$

We may expand the BW field $\Psi_{\alpha_1\alpha_2\alpha_3}$ in terms of 16 linearly independent elements as

$$\Psi_{\alpha_1\alpha_2\alpha_3}(x)=\sqrt{\tfrac{1}{2}}\left[(C)_{\alpha_1\alpha_2}\psi_{\alpha_3}(x)+(\gamma^\mu C)_{\alpha_1\alpha_2}\psi_{\alpha_3\mu}(x)+\tfrac{1}{2}(\sigma^{\mu\nu}C)_{\alpha_1\alpha_2}\psi_{\alpha_3\mu\nu}(x)\right], \tag{21}$$

with

$$\begin{aligned}\psi_{\alpha_3}(x)&=\frac{\sqrt{2}}{4}(C^{-1})^{\alpha_2\alpha_1}\Psi_{\alpha_1\alpha_2\alpha_3}(x),\\\psi_{\alpha_3\mu}(x)&=\frac{\sqrt{2}}{4}(C^{-1}\gamma_\mu)^{\alpha_2\alpha_1}\Psi_{\alpha_1\alpha_2\alpha_3}(x),\end{aligned} \tag{22}$$

and

$$\psi_{\alpha_3\mu\nu}(x)=\frac{\sqrt{2}}{4}(C^{-1}\sigma_{\mu\nu})^{\alpha_2\alpha_1}\Psi_{\alpha_1\alpha_2\alpha_3}(x).$$

The adjoint field of $\Psi^{\alpha_1\alpha_2\alpha_3}$ is denoted by $\overline{\Psi}^{\alpha_3\alpha_2\alpha_1}$ and is given by

$$\overline{\Psi}^{\alpha_3\alpha_2\alpha_1}(x)=\sqrt{\frac{1}{2}}\left[-\overline{\psi}^{\alpha_3}(x)(C^{-1})^{\alpha_2\alpha_1}+\overline{\psi}^{\alpha_3}_{\mu}(x)(C^{-1}\gamma^\mu)^{\alpha_2\alpha_1}-\overline{\psi}^{\alpha_3}_{\mu\nu}(x)\frac{1}{2}(C^{-1}\sigma^{\mu\nu})^{\alpha_2\alpha_1}\right],$$

where

$$\begin{aligned}&\overline{\psi}^{\alpha_3}(x)=-\overline{\Psi}^{\alpha_3\alpha_2\alpha_1}(x)\frac{\sqrt{2}}{4}(C)_{\alpha_1\alpha_2}=\psi^{\dagger\beta_3}(x)\xi_{\beta_3}{}^{\alpha_3},\\&\overline{\psi}^{\alpha_3}_{\mu}(x)=\overline{\Psi}^{\alpha_3\alpha_2\alpha_1}(x)\frac{\sqrt{2}}{4}(\gamma_\mu C)_{\alpha_1\alpha_2}=\psi^{\dagger\beta_3}_{\mu}(x)\xi_{\beta_3}{}^{\alpha_3},\\&\text{and}\quad\overline{\psi}^{\alpha_3}_{\mu\nu}(x)=-\overline{\Psi}^{\alpha_3\alpha_2\alpha_1}(x)\frac{\sqrt{2}}{4}(\sigma_{\mu\nu}C)_{\alpha_1\alpha_2}=\psi^{\dagger\beta_3}_{\mu\nu}(x)\xi_{\beta_3}{}^{\alpha_3}.\end{aligned} \tag{23}$$

With equation (19), we find the Euler–Lagrange equation of motion

$$\Lambda_{\alpha_1\alpha_2\alpha_3}{}^{\beta_1\beta_2\beta_3}(\partial)\Phi_{\beta_1\beta_2\beta_3}(x) = 0, \tag{24}$$

which leads to

$$\psi_{\alpha_3}(x) = 0, \tag{25}$$

$$\begin{aligned} &\mathrm{i}\partial^\nu\psi_{\alpha_3\mu\nu}(x) + \kappa_m\psi_{\alpha_3\mu}(x) + \frac{1}{\kappa_m}\left[\left(\partial^2 - \kappa_m^2\right)\psi_{\alpha_3\mu}(x) - \partial_\mu\partial^\nu\psi_{\alpha_3\nu}(x)\right] + \Lambda_{\mu\nu}(x)\psi^\nu_{\alpha_3}(x) = 0, \\ &\text{and} \quad \mathrm{i}[\partial_\mu\psi_{\alpha_3\nu}(x) - \partial_\nu\psi_{\alpha_3\mu}(x)] + \kappa_m\psi_{\alpha_3\mu\nu}(x) = 0. \end{aligned} \tag{26}$$

These equations give us

$$\Lambda_{\mu\nu}(\partial)\psi_{\alpha_3}{}^\nu(x) = 0,$$

which, in turn, allows us to write

$$(\gamma\cdot\partial + \kappa_m)\psi_{\alpha_3\mu}(x) = 0,$$

together with

$$\partial^\mu\psi_{\alpha_3\mu}(x) = \gamma\cdot\psi_{\alpha_3}(x) = 0. \tag{27}$$

From equations (21), (25), (26) and (27), we find

$$\Psi_{\alpha_1\alpha_2\alpha_3}(x) = -\frac{1}{\sqrt{2}}\frac{1}{\kappa_m}[(\gamma\cdot\partial - \kappa_m)\gamma^\mu C]_{\alpha_1\alpha_2}\psi_{\alpha_3\mu}(x), \tag{28}$$

which leads to

$$(\gamma\cdot\partial + \kappa_m)_\alpha{}^{\alpha_1}\Psi_{\alpha_1\alpha_2\alpha_3}(x) = 0,$$

where we have utilized

$$\left(\partial^2 - \kappa_m^2\right)\psi_{\alpha_3\mu}(x) = 0.$$

The symmetry of the spinor $\Psi_{\alpha_1\alpha_2\alpha_3}$ between α_1 and α_2 is clear from equation (28). If we multiply equation (28) by the skew symmetric matrix $(C^{-1})^{\alpha_2\alpha_3}$ and use equation (27), we obtain

$$\Psi_{\alpha_1\alpha_2\alpha_3}(x)(C^{-1})^{\alpha_2\alpha_3} = -\frac{1}{\sqrt{2}}\frac{1}{\kappa_m}(\gamma\cdot\partial - \kappa_m)_{\alpha_1}{}^{\alpha_3}[\gamma^\mu\psi_{\alpha_3\mu}(x)] = 0,$$

which proves that the spinor $\Psi_{\alpha_1\alpha_2\alpha_3}$ is fully symmetric.

We now proceed to find the Klein–Gordon divisor $d(\partial)$ satisfying

$$\Lambda_{\alpha_1\alpha_2\alpha_3}{}^{\gamma_1\gamma_2\gamma_3}(\partial)d_{\gamma_1\gamma_2\gamma_3}{}^{\beta_1\beta_2\beta_3}(\partial) = \left(\partial^2 - \kappa_m^2\right)I_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\delta_{\alpha_3}{}^{\beta_3}.$$

As can be verified by a tedious but straightforward calculation, we may take

$$\begin{aligned} d_{\alpha_1\alpha_2\alpha_3}{}^{\beta_1\beta_2\beta_3}(\partial) = {}& \frac{1}{\kappa_m}\left[I_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} - \frac{1}{4}(\gamma^\mu C)_{\alpha_1\alpha_2}\eta_{\mu\nu}(C^{-1}\gamma^\nu)^{\beta_2\beta_1}\right]\delta_{\alpha_3}{}^{\beta_3}\left(\partial^2 - \kappa_m^2\right) \\ & + \frac{1}{4}\left[(\gamma^\mu C)_{\alpha_1\alpha_2} + \mathrm{i}(\sigma^{\mu\kappa}C)_{\alpha_1\alpha_2}\frac{1}{\kappa_m}\partial_\kappa\right][d_{\mu\nu}(\partial)]_{\alpha_3}{}^{\beta_3} \\ & \times\left[(C^{-1}\gamma^\nu)^{\beta_2\beta_1} - \mathrm{i}\frac{1}{\kappa_m}\partial_\lambda(C^{-1}\sigma^{\nu\lambda})^{\beta_2\beta_1}\right], \end{aligned} \tag{29}$$

where

$$\begin{aligned} d_{\mu\nu}(\partial) = {}& (\gamma\cdot\partial - \kappa_m)\left[\eta_{\mu\nu} - \frac{1}{4}\gamma_\mu\gamma_\nu + \frac{1}{4}\frac{1}{\kappa_m}(\gamma_\mu\partial_\nu - \gamma_\nu\partial_\mu) - \frac{3}{4}\frac{1}{\kappa_m^2}\partial_\mu\partial_\nu\right] \\ & + \frac{3}{4}\frac{1}{\kappa_m^2}\left(\partial^2 - \kappa_m^2\right)[(\gamma_\mu\partial_\nu - \gamma_\nu\partial_\mu) + (\gamma\cdot\partial - \kappa_m)\gamma_\mu\gamma_\nu], \end{aligned}$$

which satisfies

$$\Lambda_{\mu\lambda}(\partial)d^{\lambda}{}_{\nu}(\partial) = \left(\partial^2 - \kappa_m^2\right)\eta_{\mu\nu}.$$

Here we have used the Fierz identity (A.2).

We can quantize the BW field Ψ by using the Klein–Gordon divisor [9]

$$\left\{\Psi_{\alpha_1\alpha_2\alpha_3}(x), \overline{\Psi}^{\beta_3\beta_2\beta_1}(y)\right\} = -\mathrm{i}d_{\alpha_1\alpha_2\alpha_3}{}^{\beta_1\beta_2\beta_3}(\partial)\Delta(x-y,\kappa_m),$$
$$\text{and}\quad \langle 0|T(\Psi_{\alpha_1\alpha_2\alpha_3}(x)\overline{\Psi}^{\beta_3\beta_2\beta_1}(y))|0\rangle = -\mathrm{i}d_{\alpha_1\alpha_2\alpha_3}{}^{\beta_1\beta_2\beta_3}(\partial)\Delta_C(x-y,\kappa_m), \tag{30}$$

where Δ and Δ_C are defined in equations (17) and (18), respectively. By using equations (22) and (23), we obtain from equation (30):

$$\left\{\psi_{\alpha_3\mu}(x), \overline{\psi}_{\nu}{}^{\beta_3}(y)\right\} = -\mathrm{i}[d_{\mu\nu}(\partial)]_{\alpha_3}{}^{\beta_3}\Delta(x-y,\kappa_m).$$

The charge conjugation operator C can be eliminated from equations (20) and (29) with the aid of the Fierz identities in appendix A. However, the final expressions, given in appendix B, are quite complicated.

4. BW field with arbitrary spin

According to the BW method [10], a field which corresponds to a spin-S particle is described by a multi-spinor wavefunction $\psi_{\alpha_1\cdots\alpha_n}(x)$, where $n = 2S$, and ψ is assumed to be completely symmetric with respect to the indices $\alpha_1, \ldots, \alpha_n$. In NQM, Hagen and Hurley have described it with a multi-spinor field with arbitrary spin S which transforms as a direct product of n spinors of spin $S = 1/2$ under Galilean transformations [1]. Following Hagen and Hurley, we assume here that a BW field can be decomposed into the direct product of N symmetric spinor fields of rank 2, for S even, and the direct product of N symmetric spinor fields of rank 2 with one symmetric spinor field of rank 3, for S odd, where $N = [n/2]$. We also introduce N sets of 4×4 Galilean gamma matrices $\gamma^{(k)\mu}$, together with the charge conjugation matrix $C^{(k)} = -C^{(k)T}$, where k runs from 1 to N, which acts only on the kth double spinor indices α_{2k-1} and α_{2k}, for S even. In the case of $k = N$ for S odd, the gamma matrices and conjugation matrix act on the Nth triple spinor indices $\alpha_{n-2}, \alpha_{n-1}$ and α_n.

In the following subsections, we discuss separately the n even and n odd situations.

4.1. BW fields with n even

A completely symmetric BW field is given by

$$\Phi_{\alpha_1\cdots\alpha_n}(x) = \left(\frac{1}{m}\right)^{3(N-1)/2}\sum_C\prod_{k=1}^{N}\Phi^{(k)}_{\alpha_{2k-1}\alpha_{2k}}(x),$$

where $\sum_C$ denotes the sum over all distinct combinations of indices $\alpha_1, \ldots, \alpha_n$, and

$$\Phi^{(k)}_{\alpha_{2k-1}\alpha_{2k}}(x) = \sqrt{\frac{m}{2}}\bigg[(C^{(k)})_{\alpha_{2k-1}\alpha_{2k}}\varphi^{(k)}(x) + (\gamma^{(k)\mu}C^{(k)})_{\alpha_{2k-1}\alpha_{2k}}\varphi^{(k)}_{\mu}(x)$$
$$+\frac{1}{2}(\sigma^{(k)\mu\nu}C^{(k)})_{\alpha_{2k-1}\alpha_{2k}}\varphi^{(k)}_{\mu\nu}(x)\bigg]. \tag{31}$$

The adjoint field to Φ is written as $\overline{\Phi}$ and is given by

$$\overline{\Phi}^{\alpha_n\cdots\alpha_1}(x) = \left(\frac{1}{m}\right)^{3(N-1)/2}\sum_C\prod_{k=1}^{N-1}\overline{\Phi}^{(N-k)\alpha_{2(N-k)}\alpha_{2(N-k)-1}}(x),$$

where

$$\overline{\Phi}^{(N-k)\alpha_{2(N-k)}\alpha_{2(N-k)-1}}(x) = \Phi^{(N-k)\dagger\beta_{2(N-k)}\beta_{2(N-k)-1}}(x)(\xi^{(N-k)})_{\beta_{2(N-k)-1}}{}^{\alpha_{2(N-k)-1}}(\xi^{(N-k)})_{\beta_{2(N-k)}}{}^{\alpha_{2(N-k)}}$$

$$= \sqrt{\frac{m}{2}}\Big[-\overline{\varphi}^{(N-k)}(x)(C^{(N-k)-1})^{\alpha_{2(N-k)}\alpha_{2(N-k)-1}} + \overline{\varphi}^{(N-k)\mu}(x)\big(C^{(N-k)-1}\gamma_\mu^{(N-k)}\big)^{\alpha_{2(N-k)}\alpha_{2(N-k)-1}}$$

$$- \overline{\varphi}^{(N-k)\mu\nu}(x)\frac{1}{2}\big(C^{(N-k)-1}\sigma_{\mu\nu}^{(N-k)}\big)^{\alpha_{2(N-k)}\alpha_{2(N-k)-1}}\Big],$$

with

$$\overline{\varphi}^{(N-k)}(x) = \varphi^{(N-k)\dagger}(x),$$
$$\overline{\varphi}^{(N-k)\mu}(x) = \eta^{\mu\nu}\varphi_\nu^{(N-k)\dagger}(x),$$
$$\text{and}\quad \overline{\varphi}^{(N-k)\mu\nu}(x) = \eta^{\mu\rho}\eta^{\nu\sigma}\varphi_{\rho\sigma}^{(N-k)\dagger}(x).$$

Let us write the corresponding Lagrangian in the form

$$L_{\text{even}}(x) = -\overline{\Phi}^{\alpha_n\cdots\alpha_1}(x)\Lambda^{(e)}_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}(\partial)\Phi_{\beta_1\cdots\beta_n}(x), \tag{32}$$

where

$$\Lambda^{(e)}_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}(\partial) = \sum_{k=1}^{N}\Lambda^{(k)}_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}}(\partial)\prod_{l=1,l\neq k}^{N} I^{(l)}_{\alpha_{2l-1}\alpha_{2l}}{}^{\beta_{2l-1}\beta_{2l}}, \tag{33}$$

with

$$\Lambda_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}}(\partial) = \frac{1}{\sqrt{2}}\big[(\Gamma^{(k)\mu})_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}}\partial_\mu + \kappa_m I^{(k)}_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}}\big]. \tag{34}$$

The matrices $I^{(k)}$ and $\Gamma^{(k)\mu}$ are defined as

$$I^{(k)}_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}} = \tfrac{1}{4}\big[(C^{(k)})_{\alpha_{2k-1}\alpha_{2k}}(C^{(k)-1})^{\beta_{2k}\beta_{2k-1}} + (\gamma^{(k)\mu}C^{(k)})_{\alpha_{2k-1}\alpha_{2k}}\big(C^{(k)-1}\gamma_\mu^{(k)}\big)^{\beta_{2k}\beta_{2k-1}}$$
$$+ \tfrac{1}{2}(\sigma^{(k)\mu\nu}C^{(k)})_{\alpha_{2k-1}\alpha_{2k}}\big(C^{(k)-1}\sigma_{\mu\nu}^{(k)}\big)^{\beta_{2k}\beta_{2k-1}}\big],$$
$$= \delta^{(k)}_{\alpha_{2k-1}}{}^{\beta_{2k-1}}\delta^{(k)}_{\alpha_{2k}}{}^{\beta_{2k}}$$

and

$$(\Gamma^{(k)\mu})_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}} = \frac{\mathrm{i}}{4}\big[(\sigma^{(k)\mu\nu}C^{(k)})_{\alpha_{2k-1}\alpha_{2k}}\big(C^{(k)-1}\gamma_\nu^{(k)}\big)^{\beta_{2k}\beta_{2k-1}}$$
$$- \big(\gamma_\nu^{(k)}C^{(k)}\big)_{\alpha_{2k-1}\alpha_{2k}}(C^{(k)-1}\sigma^{(k)\mu\nu})^{\beta_{2k}\beta_{2k-1}}\big],$$
$$= \frac{1}{4}\big[(\gamma^{(k)\mu})_{\alpha_{2k-1}}{}^{\beta_{2k-1}}\delta^{(k)\beta_{2k}}_{\alpha_{2k}} + (\gamma^{(k)\mu})_{\alpha_{2k-1}}{}^{\beta_{2k}}\delta^{(k)\beta_{2k-1}}_{\alpha_{2k}}$$
$$+ \delta^{(k)}_{\alpha_{2k-1}}{}^{\beta_{2k}}(\gamma^{(k)\mu})_{\alpha_{2k}}{}^{\beta_{2k-1}} + \delta^{(k)}_{\alpha_{2k-1}}{}^{\beta_{2k-1}}(\gamma^{(k)\mu})_{\alpha_{2k}}{}^{\beta_{2k}}\big].$$

The Euler–Lagrange equation of motion follows immediately from equation (32):

$$\sum_C \Lambda^{(e)}_{\alpha_1\cdots\alpha_n}{}^{\gamma_1\cdots\gamma_n}(\partial)\Phi_{\gamma_1\cdots\gamma_n}(x) = 0.$$

Multiplying this equation with the identity operator defined by

$$I^{(e)}_{\beta_1\cdots\beta_n}{}^{\alpha_1\cdots\alpha_n} = \prod_{k=1}^{N} I^{(k)}_{\beta_{2k-1}\beta_{2k}}{}^{\alpha_{2k-1}\alpha_{2k}}, \tag{35}$$

we obtain, after some calculations,

$$\Lambda^{(k)}_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}}(\partial)\Phi^{(k)}_{\beta_{2k-1}\beta_{2k}}(x) = 0, \tag{36}$$

where $k = 1, \ldots, N$. This equation yields

$$\varphi^{(k)}(x) = 0, \tag{37}$$

$$\begin{aligned} &\varphi^{(k)}_{\mu\nu}(x) = \mathrm{i}\frac{1}{\kappa_m}\big[\partial_\nu\varphi^{(k)}_\mu(x) - \partial_\mu\varphi^{(k)}_\nu(x)\big], \\ &\text{and} \quad \big(\partial^2\delta^{(k)\nu}_\mu - \partial_\mu\partial^\nu - \kappa_m^2\delta^{(k)\nu}_\mu\big)\varphi^{(k)}_\nu(x) = 0, \end{aligned} \tag{38}$$

which leads to

$$\partial^\mu\varphi^{(k)}_\mu(x) = 0,$$

and

$$\big(\partial^2 - \kappa_m^2\big)\varphi^{(k)}_\mu(x) = 0.$$

If we substitute equations (37) and (38) back into equation (31), we obtain

$$\Phi^{(k)}_{\alpha_{2k-1}\alpha_{2k}}(x) = -\sqrt{\frac{m}{2}}\frac{1}{\kappa_m}[(\gamma^{(k)}\cdot\partial - \kappa_m)\gamma^{(k)\mu}C^{(k)}]_{\alpha_{2k-1}\alpha_{2k}}\varphi^{(k)}_\mu(x),$$

from which we have

$$\Phi^{(k)}_{\alpha_{2k-1}\alpha_{2k}}(x) = \Phi^{(k)}_{\alpha_{2k}\alpha_{2k-1}}(x),$$

and

$$(\gamma^{(k)}\cdot\partial + \kappa_m)_\alpha{}^{\alpha_{2k-1}}\Phi^{(k)}_{\alpha_{2k-1}\alpha_{2k}}(x) = 0.$$

Recalling equation (35) and introducing a field ϕ given by

$$\phi_{\alpha_1\cdots\alpha_n}(x) = I^{(e)}_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}\Phi_{\beta_1\cdots\beta_n}(x),$$

we find

$$\phi_{\alpha_1\cdots\alpha_n}(x) = \left(\frac{1}{m}\right)^{3(N-1)/2}\prod_{k=1}^{N}\Phi^{(k)}_{\alpha_{2k-1}\alpha_{2k}}(x).$$

We therefore obtain

$$\begin{aligned} &\Phi_{\alpha_1\cdots\alpha_n}(x) = \sum_C \phi_{\alpha_1\cdots\alpha_n}(x), \\ &\text{and} \quad (\gamma^{(k)}\cdot\partial + \kappa_m)_\alpha{}^{\alpha_{2k-1}}\Phi_{\alpha_1\cdots\alpha_{2k-1}\cdots\alpha_n}(x) = 0. \end{aligned} \tag{39}$$

The operator $d^{(e)}$, reciprocal to $\Lambda^{(e)}$, satisfies

$$\Lambda^{(e)}_{\alpha_1\cdots\alpha_n}{}^{\gamma_1\cdots\gamma_n}(\partial)d^{(e)}_{\gamma_1\cdots\gamma_n}{}^{\beta_1\cdots\beta_n}(\partial) = I^{(e)}_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}\big(\partial^2 - \kappa_m^2\big),$$

where $\Lambda^{(e)}$ is given by equation (33). We may write it in the form

$$d^{(e)}_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}(\partial) = \sum_{k=1}^{N} d^{(k)}_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}}(\partial)\prod_{l=1,l\neq k}^{N} I^{(l)}_{\alpha_{2l-1}\alpha_{2l}}{}^{\beta_{2l-1}\beta_{2l}}, \tag{40}$$

with

$$\begin{aligned} d^{(k)}_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}}(\partial) = \sqrt{2}\bigg[&\frac{1}{\kappa_m}\big(\partial^2 - \kappa_m^2\big)I^{(k)}_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}} + (\Gamma^{(k)\mu})_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}}\partial_\mu \\ &- \frac{1}{\kappa_m}(\Gamma^{(k)\mu})_{\alpha_{2k-1}\alpha_{2k}}{}^{\gamma_{2k-1}\gamma_{2k}}\partial_\mu(\Gamma^{(k)\nu})_{\gamma_{2k-1}\gamma_{2k}}{}^{\beta_{2k-1}\beta_{2k}}\partial_\nu\bigg]. \end{aligned} \tag{41}$$

The covariant commutation relation and the causal Green's function are given by using the Klein–Gordon divisor of equation (40):

$$\begin{aligned} &\big[\phi_{\alpha_1\cdots\alpha_n}(x), \overline{\phi}^{\beta_n\cdots\beta_1}(y)\big] = -\mathrm{i}d^{(e)}_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}(\partial)\Delta(x-y,\kappa_m), \\ &\langle 0|T(\phi_{\alpha_1\cdots\alpha_n}(x)\overline{\phi}^{\beta_n\cdots\beta_1}(y))|0\rangle = -\mathrm{i}d^{(e)}_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}(\partial)\Delta_C(x-y,\kappa_m). \end{aligned}$$

Thus we find

$$\left[\Phi_{\alpha_1\cdots\alpha_n}(x), \overline{\Phi}^{\beta_n\cdots\beta_1}(y)\right] = \sum_C \left[\phi_{\alpha_1\cdots\alpha_n}(x), \overline{\phi}^{\beta_n\cdots\beta_1}(y)\right],$$

where $\sum_C$ means the sum over all distinct combinations of the indices $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$.

4.2. BW fields with n odd

A completely symmetric BW field is given as

$$\Psi_{\alpha_1\cdots\alpha_n}(x) = \left(\frac{1}{m}\right)^{3(N-1)/2} \sum_C \prod_{k=1}^{N-1} \Phi^{(k)}_{\alpha_{2k-1}\alpha_{2k}}(x) \Psi^{(N)}_{\alpha_{n-2}\alpha_{n-1}\alpha_n}(x), \tag{42}$$

where

$$\Psi^{(N)}_{\alpha_{n-2}\alpha_{n-1}\alpha_n}(x) = \sqrt{\frac{1}{2}}\Big[(C^{(N)})_{\alpha_{n-2}\alpha_{n-1}}\psi^{(N)}_{\alpha_n}(x) + (\gamma^{(N)\mu}C^{(N)})_{\alpha_{n-2}\alpha_{n-1}}\psi^{(N)}_{\alpha_n\mu}(x) + \frac{1}{2}(\sigma^{(N)\mu\nu}C^{(N)})_{\alpha_{n-2}\alpha_{n-1}}\psi^{(N)}_{\alpha_n\mu\nu}(x)\Big], \tag{43}$$

we find that the main difference with the argument of section 4.1 follows from the replacement of $\Phi^{(N)}$ by $\Psi^{(N)}$. So far as $\Phi^{(k)}$ $(k = 1, \ldots, N-1)$ is concerned, the results are the same as those in the previous subsection. Henceforth, we shall discuss parts which concern $\Psi^{(N)}$.

The adjoint field to Ψ, denotes by $\overline{\Psi}$, is expressed by

$$\overline{\Psi}^{\alpha_n\cdots\alpha_1}(x) = \left(\frac{1}{m}\right)^{3(N-1)/2} \sum_C \overline{\Psi}^{(N)\alpha_n\alpha_{n-1}\alpha_{n-2}}(x) \prod_{k=1}^{N-1} \overline{\Phi}^{(N-k)\alpha_{2(N-k)}\alpha_{2(N-k)-1}}(x),$$

where

$$\begin{aligned}\overline{\Psi}^{(N)\alpha_n\alpha_{n-1}\alpha_{n-2}}(x) &= \Psi^{(N)\dagger\beta_n\beta_{n-1}\beta_{n-2}}(x)(\xi^{(N)})_{\beta_{n-2}}{}^{\alpha_{n-2}}(\xi^{(N)})_{\beta_{n-1}}{}^{\alpha_{n-1}}(\xi^{(N)})_{\beta_n}{}^{\alpha_n}\\ &= \sqrt{\frac{1}{2}}\Big[-\overline{\psi}^{(N)\alpha_n}(x)(C^{(N)-1})^{\alpha_{n-1}\alpha_{n-2}} + \overline{\psi}^{(N)\alpha_n}_{\mu}(x)(C^{(N)-1}\gamma^{(N)\mu})^{\alpha_{n-1}\alpha_{n-2}}\\ &\quad - \frac{1}{2}\overline{\psi}^{(N)\alpha_n}_{\mu\nu}(x)(C^{(N)-1}\sigma^{(N)\mu\nu})^{\alpha_{n-1}\alpha_{n-2}}\Big],\end{aligned}$$

with

$$\overline{\psi}^{(N)\alpha_n}(x) = \psi^{(N)\dagger\beta_n}(x)(\xi^{(N)})_{\beta_n}{}^{\alpha_n},$$

$$\overline{\psi}^{(N)\alpha_n}_{\mu}(x) = \psi^{(N)\dagger\beta_n}_{\mu}(x)(\xi^{(N)})_{\beta_n}{}^{\alpha_n},$$

and $\overline{\psi}^{(N)\alpha_n}_{\mu\nu}(x) = \psi^{(N)\dagger\beta_n}_{\mu\nu}(x)(\xi^{(N)})_{\beta_n}{}^{\alpha_n}$.

We write the Lagrangian as follows:

$$L_{\text{odd}}(x) = -\overline{\Psi}^{\alpha_n\cdots\alpha_1}(x)\Lambda^{(o)}_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}(\partial)\Psi_{\beta_1\cdots\beta_n}(x), \tag{44}$$

where

$$\Lambda^{(o)}_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}(\partial) = \sum_{k=1}^{N-1} \Lambda^{(k)}_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}}(\partial) \prod_{l=1,l\neq k}^{N} I^{(l)}_{\alpha_{2l-1}\alpha_{2l}}{}^{\beta_{2l-1}\beta_{2l}}\delta^{(N)\beta_n}_{\alpha_n} + \prod_{k=1}^{N-1} I^{(k)}_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}}\Lambda^{(N)}_{\alpha_{n-2}\alpha_{n-1}\alpha_n}{}^{\beta_{n-2}\beta_{n-1}\beta_n}(\partial). \tag{45}$$

Here, $\Lambda^{(k)}(\partial)$ ($k = 1, \ldots, N-1$) is given by equation (34) and

$$\Lambda^{(N)}_{\alpha_{n-2}\alpha_{n-1}\alpha_n}{}^{\beta_{n-2}\beta_{n-1}\beta_n}(\partial) = \Big\{(\Gamma^{(N)\mu})_{\alpha_{n-2}\alpha_{n-1}}{}^{\beta_{n-2}\beta_{n-1}}\partial_\mu + \kappa_m I^{(N)}_{\alpha_{n-2}\alpha_{n-1}}{}^{\beta_{n-2}\beta_{n-1}}$$
$$+ \frac{1}{\kappa_m}\frac{1}{4}(\gamma^{(N)\mu}C^{(N)})_{\alpha_{n-2}\alpha_{n-1}}\big[(\partial^2 - \kappa_m^2)\eta_{\mu\nu} - \partial_\mu\partial_\nu\big]$$
$$\times (C^{(N)-1}\gamma^{(N)\nu})^{\beta_{n-1}\beta_{n-2}}\Big\}\delta^{(N)\beta_n}_{\alpha_n}$$
$$+ \frac{1}{4}(\gamma^{(N)\mu}C^{(N)})_{\alpha_{n-2}\alpha_{n-1}}\big[\Lambda^{(N)}_{\mu\nu}(\partial)\big]_{\alpha_n}{}^{\beta_n}(C^{(N)-1}\gamma^{(N)\nu})^{\beta_{n-1}\beta_{n-2}},$$

where

$$\Lambda^{(N)}_{\mu\nu}(\partial) = (\gamma^{(N)}\cdot\partial + \kappa_m)\left(\eta_{\mu\nu} - \frac{1}{4}\gamma^{(N)}_\mu\gamma^{(N)}_\nu\right) - \frac{1}{4}\big(\gamma^{(N)}_\mu\partial_\nu - \gamma^{(N)}_\nu\partial_\mu\big).$$

The Euler–Lagrange equation of motion comes from equation (44):

$$\sum_C \Lambda^{(o)}_{\alpha_1\cdots\alpha_n}{}^{\gamma_1\cdots\gamma_n}(\partial)\Psi_{\gamma_1\cdots\gamma_n}(x) = 0.$$

If we multiply this equation by the operator

$$I^{(o)}_{\beta_1\cdots\beta_n}{}^{\alpha_1\cdots\alpha_n} = \prod_{k=1}^{N} I^{(k)}_{\beta_{2k-1}\beta_{2k}}{}^{\alpha_{2k-1}\alpha_{2k}}\delta^{(N)\beta_n}_{\alpha_n},$$

we arrive at equation (36), for $k = 1, \ldots, N-1$, as well as

$$\Lambda^{(N)\beta_{n-2}\beta_{n-1}\beta_n}_{\alpha_{n-2}\alpha_{n-1}\alpha_n}(\partial)\Psi^{(N)}_{\beta_{n-2}\beta_{n-1}\beta_n}(x) = 0.$$

These equations lead to

$$\psi^{(N)}_{\alpha_n}(x) = 0, \tag{46}$$

$$\psi^{(N)}_{\alpha_n\mu\nu}(x) = \mathrm{i}\frac{1}{\kappa_m}\big[\partial_\nu\psi^{(N)}_{\alpha_n\mu}(x) - \partial_\mu\psi^{(N)}_{\alpha_n\nu}(x)\big], \tag{47}$$

$$\text{and}\quad \big[\Lambda^{(N)}_{\mu\nu}(\partial)\big]_{\alpha_n}{}^{\beta_n}\psi^{(N)\nu}_{\beta_n}(x) = 0. \tag{48}$$

Then from equation (48) we get

$$(\gamma^{(N)\mu})_{\alpha_n}{}^{\beta_n}\psi^{(N)}_{\beta_n\mu}(x) = 0,$$
$$\partial^\mu\psi^{(N)}_{\beta_n\mu}(x) = 0,$$
$$\text{and}\quad (\gamma^{(N)}\cdot\partial + \kappa_m)_{\alpha_n}{}^{\beta_n}\psi^{(N)}_{\beta_n\mu}(x) = 0.$$

By substituting equations (46) and (47) back into equation (43), we get

$$\Psi^{(N)}_{\alpha_{n-2}\alpha_{n-1}\alpha_n}(x) = -\sqrt{\frac{1}{2}}\frac{1}{\kappa_m}[(\gamma^{(N)}\cdot\partial - \kappa_m)\gamma^{(N)\mu}C^{(N)}]_{\alpha_{n-2}\alpha_{n-1}}\psi^{(N)}_{\alpha_n\mu}(x).$$

Note that the symmetry property of $\Psi^{(N)}$ with respect to the indices α_{n-2}, α_{n-1} and α_n follows from this equation, and

$$\Psi^{(N)}_{\alpha_{n-2}\alpha_{n-1}\alpha_n}(x)(C^{(N)-1})^{\alpha_{n-1}\alpha_n} = 0.$$

In terms of the field

$$\psi_{\alpha_1\cdots\alpha_n}(x) = I^{(o)}_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}\Psi_{\beta_1\cdots\beta_n}(x),$$
$$= \left(\frac{1}{m}\right)^{3(N-1)/2}\prod_{k=1}^{N-1}\Phi^{(k)}_{\alpha_{2k-1}\alpha_{2k}}(x)\Psi^{(N)}_{\alpha_{n-2}\alpha_{n-1}\alpha_n}(x),$$

we can express equation (42) as

$$\Psi_{\alpha_1\cdots\alpha_n}(x) = \sum_C \psi_{\alpha_1\cdots\alpha_n}(x).$$

Hence, we find

$$(\gamma^{(k)}\cdot\partial + \kappa_m)_\alpha{}^{\alpha_{2k-1}}\Psi_{\alpha_1\cdots\alpha_{2k-1}\cdots\alpha_n}(x) = 0. \tag{49}$$

We can write the Klein–Gordon divisor $d^{(o)}(\partial)$ in the form

$$d^{(o)}_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}(\partial) = \sum_{k=1}^{N-1} d^{(k)}_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}}(\partial)\prod_{l=1,l\neq k}^{N} I^{(l)}_{\alpha_{2l-1}\alpha_{2l}}{}^{\beta_{2l-1}\beta_{2l}}\delta^{(N)\beta_n}_{\alpha_n}$$
$$+\prod_{k=1}^{N-1} I^{(k)}_{\alpha_{2k-1}\alpha_{2k}}{}^{\beta_{2k-1}\beta_{2k}} d^{(N)}_{\alpha_{n-2}\alpha_{n-1}\alpha_n}{}^{\beta_{n-2}\beta_{n-1}\beta_n}(\partial),$$

where $d^{(k)}(\partial)$, $k = 1, \ldots, N-1$, is given by equation (41) and

$$d^{(N)}_{\alpha_{n-2}\alpha_{n-1}\alpha_n}{}^{\beta_{n-2}\beta_{n-1}\beta_n}(\partial) = \frac{1}{\kappa_m}\left(\partial^2-\kappa_m^2\right)\Big[I^{(N)}_{\alpha_{n-2}\alpha_{n-1}}{}^{\beta_{n-2}\beta_{n-1}}$$
$$-\frac{1}{4}(\gamma^{(N)\mu}C^{(N)})_{\alpha_{n-2}\alpha_{n-1}}\left(C^{(N)-1}\gamma^{(N)}_\mu\right)^{\beta_{n-1}\beta_{n-2}}\Big]\delta^{(N)\beta_n}_{\alpha_n}$$
$$+\frac{1}{4}\left[(\gamma^{(N)\mu}C^{(N)})_{\alpha_{n-2}\alpha_{n-1}} + \mathrm{i}(\sigma^{(N)\mu\kappa}C^{(N)})_{\alpha_{n-2}\alpha_{n-1}}\frac{1}{\kappa_m}\partial_\kappa\right]\left[d^{(N)}_{\mu\nu}(\partial)\right]_{\alpha_n}{}^{\beta_n}$$
$$\times\left[(C^{(N)-1}\gamma^{(N)\nu})^{\beta_{n-1}\beta_{n-2}} - \mathrm{i}\frac{1}{\kappa_m}\partial_\lambda(C^{(N)-1}\sigma^{(N)\nu\lambda})^{\beta_{n-1}\beta_{n-2}}\right],$$

with

$$d^{(N)}_{\mu\nu}(\partial) = (\gamma^{(N)}\cdot\partial-\kappa_m)\left[\eta_{\mu\nu}-\frac{1}{4}\gamma^{(N)}_\mu\gamma^{(N)}_\nu+\frac{1}{4}\frac{1}{\kappa_m}\left(\gamma^{(N)}_\mu\partial_\nu-\gamma^{(N)}_\nu\partial_\mu\right)-\frac{3}{4}\frac{1}{\kappa_m^2}\partial_\mu\partial_\nu\right]$$
$$+\frac{3}{4}\frac{1}{\kappa_m^2}\left(\partial^2-\kappa_m^2\right)\left[\gamma^{(N)}_\mu\partial_\nu-\gamma^{(N)}_\nu\partial_\mu+(\gamma^{(N)}\cdot\partial-\kappa_m)\gamma^{(N)}_\mu\gamma^{(N)}_\nu\right].$$

After a lengthy calculation, we verify that the Klein–Gordon divisor satisfies

$$\Lambda^{(o)}_{\alpha_1\cdots\alpha_n}{}^{\gamma_1\cdots\gamma_n}(\partial)d^{(o)}_{\gamma_1\cdots\gamma_n}{}^{\beta_1\cdots\beta_n}(\partial) = I^{(o)}_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}\left(\partial^2-\kappa_m^2\right).$$

Thus we find

$$\left\{\psi_{\alpha_1\cdots\alpha_n}(x),\overline{\psi}^{\beta_n\cdots\beta_1}(y)\right\} = -\mathrm{i}d^{(o)}_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}(\partial)]\Delta(x-y,\kappa_m),$$
$$\text{and}\quad \langle 0|T\left(\psi_{\alpha_1\cdots\alpha_n}(x)\overline{\psi}^{\beta_n\cdots\beta_1}(y)\right)|0\rangle = -\mathrm{i}d^{(o)}_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}(\partial)\Delta_C(x-y,\kappa_m),$$

so that

$$\left\{\Psi_{\alpha_1\cdots\alpha_n}(x),\overline{\Psi}^{\beta_n\cdots\beta_1}(y)\right\} = \sum_C\left\{\psi_{\alpha_1\cdots\alpha_n}(x),\overline{\psi}^{\beta_n\cdots\beta_1}(y)\right\}.$$

5. Electromagnetic interaction

The minimal electromagnetic interaction is introduced by replacing ∂_μ with the covariant derivative

$$D_\mu = \partial_\mu - \mathrm{i}eA_\mu(x), \tag{50}$$

where a five-dimensional electromagnetic potential may be taken as

$$A^{\mu}(x) = (\mathbf{A}(\mathbf{x}, x_4), 0, \phi(\mathbf{x}, x_4)),$$

so that the covariant derivative reads

$$D_{\mu} = (\boldsymbol{\nabla} - \mathrm{i}e\mathbf{A}, \partial_4 + \mathrm{i}e\phi, \partial_5). \tag{51}$$

It is well known that a set of BW equations leads to inconsistency among themselves when the minimal electromagnetic coupling is switched on. Let us explain this situation for a symmetric rank-2 spinor field. The BW equations are then

$$(\gamma \cdot \partial + \kappa_m)_{\alpha}{}^{\alpha_1} \Phi_{\alpha_1\alpha_2}(x) = 0,$$
$$\text{and} \quad (\gamma \cdot \partial + \kappa_m)_{\beta}{}^{\alpha_2} \Phi_{\alpha_1\alpha_2}(x) = 0.$$

In the absence of electromagnetic field, the two equations are equivalent due to the symmetry property of the field. When we replace ∂ by D in these two equations, we obtain

$$(\gamma \cdot D + \kappa_m)_{\alpha}{}^{\alpha_1} \Phi_{\alpha_1\alpha_2}(x) = 0, \tag{52}$$

$$\text{and} \quad (\gamma \cdot D + \kappa_m)_{\beta}{}^{\alpha_2} \Phi_{\alpha_1\alpha_2}(x) = 0. \tag{53}$$

These two equations are inconsistent since

$$\left[(\gamma \cdot D + \kappa_m)_{\alpha}{}^{\alpha_1}(\gamma \cdot D + \kappa_m)_{\beta}{}^{\alpha_2} - (\gamma \cdot D + \kappa_m)_{\beta}{}^{\alpha_2}(\gamma \cdot D + \kappa_m)_{\alpha}{}^{\alpha_1}\right]\Phi_{\alpha_1\alpha_2}(x)$$
$$= -\mathrm{i}e[\partial_{\mu}A_{\nu}(x) - \partial_{\nu}A_{\mu}(x)](\gamma^{\mu})_{\alpha}{}^{\alpha_1}(\gamma^{\nu})_{\beta}{}^{\alpha_2}\Phi_{\alpha_1\alpha_2}(x),$$

which is not identically zero, unless $\Phi_{\alpha_1\alpha_2}$ vanishes identically. Hence, equations (52) and (53) are inconsistent.

In our Lagrange formulation, the equation of motion is

$$\Lambda_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}(\partial)\Phi_{\beta_1\beta_2}(x) = 0.$$

We may replace ∂_{μ} by D_{μ} to obtain

$$\Lambda_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}(D)\Phi_{\beta_1\beta_2}(x) = 0,$$

which turns out to be

$$\Phi_{\alpha_1\alpha_2}(x) = -\sqrt{\frac{m}{2}}\frac{1}{\kappa_m}[(\gamma \cdot D - \kappa_m)\gamma^{\mu}C]_{\alpha_1\alpha_2}\varphi_{\mu}(x),$$

with

$$\left(D^2\delta_{\mu}{}^{\nu} - D_{\mu}D^{\nu} - \kappa_m^2\delta_{\mu}{}^{\nu}\right)\varphi_{\nu}(x) = 0.$$

It is now obvious that $\Phi_{\alpha_1\alpha_2}$ does not satisfy the BW equation in which ∂_{μ} is replaced by D_{μ}, that is,

$$(\gamma \cdot D + \kappa_m)_{\alpha}{}^{\alpha_1}\Phi_{\alpha_1\alpha_2}(x) \neq 0.$$

We therefore avoid the difficulty predicted above.

The same situation occurs for symmetric rank-3 spinor field [3].

Our Lagrangian formulation for a BW field with arbitrary spin is constructed based on the symmetric spinor fields of rank 2 and rank 3. Therefore, our theory is free from the inconsistency when introducing the minimal electromagnetic coupling.

6. Schrödinger equation

First, let us consider the BW field equation (39) with n even:

$$(\gamma^{(k)} \cdot \partial + \kappa_m)_\alpha{}^{\alpha_{2k-1}} \Phi_{\alpha_1 \cdots \alpha_{2k-1} \cdots \alpha_n}(x) = 0, \tag{54}$$

with $k = 1, \ldots, N$. This equation also holds when we replace α_{2k-1} by α_{2k}, because $\gamma^{(k)}$'s act only on the pair of indices α_{2k-1} and α_{2k}. Hence, we have a set of n equations, where $n = 2S$ and $N = [n/2] = S$.

In view of the structure of $\gamma^{(k)}$'s given in equation (6), it is convenient to write down equation (54) by using 2-component multi-spinor fields. If we write the spinor indices

$$u = 1, 2, \qquad v = 3, 4,$$

so that $u = v - 2$, then we find

$$\Phi_{\alpha_1 \cdots v \cdots \alpha_n}(x) = -\mathrm{i}\frac{1}{\kappa_m}\left[(\boldsymbol{\sigma}^{(k)} \cdot \nabla)_u{}^{u_{2k-1}} + \kappa_m \delta_u^{(k)u_{2k-1}}\right]\Phi_{\alpha_1 \cdots u_{2k-1} \cdots \alpha_n}(x),$$

and (55)

$$\partial_t \Phi_{\alpha_1 \cdots u \cdots \alpha_n}(x) = \sqrt{\frac{1}{2}}\left[-(\boldsymbol{\sigma}^{(k)} \cdot \nabla)_v{}^{v_{2k-1}} + \kappa_m \delta_v^{(k)v_{2k-1}}\right]\Phi_{\alpha_1 \cdots v_{2k-1} \cdots \alpha_n}(x).$$

By combining this last equation with equation (55), we obtain

$$\mathrm{i}\partial_t \Phi_{\alpha_1 \cdots u \cdots \alpha_n}(x) = \left[-\frac{1}{2m}(\boldsymbol{\sigma}^{(k)} \cdot \nabla)^2_u{}^{u_{2k-1}} + m\delta_u^{(k)u_{2k-1}}\right]\Phi_{\alpha_1 \cdots u_{2k-1} \cdots \alpha_n}(x), \tag{56}$$

where equation (5) has been utilized. Relation (55) implies that among the components of Φ, only $\Phi_{u_1 \cdots u_n}(x)$'s are independent, so that the number of such components is $n + 1 = 2S + 1$, which is the correct number to describe a spin-S field. This allows us to rewrite equation (56) as

$$\mathrm{i}\partial_t \Phi_{u_1 \cdots u \cdots u_n}(x) = \left[-\frac{1}{2m}(\boldsymbol{\sigma}^{(k)} \cdot \nabla)^2_u{}^{u_{2k-1}} + m\delta_u^{(k)u_{2k-1}}\right]\Phi_{u_1 \cdots u_{2k-1} \cdots u_n}(x),$$

which leads to the Schrödinger equation, after summation over k and division by n:

$$\mathrm{i}\partial_t \Phi_{u_1 \cdots u \cdots u_n}(x) = (H)_u{}^{u'}\Phi_{u_1 \cdots u' \cdots u_n}(x), \tag{57}$$

with the Hamiltonian

$$H = -\frac{1}{2m}\frac{1}{2S}\left[2\sum_{k=1}^{N}(\boldsymbol{\sigma}^{(k)} \cdot \nabla)^2\right] + m.$$

The minimal electromagnetic interaction is introduced by the replacement defined in equation (50), together with the covariant derivative (equation (51)) in the Schrödinger equation (57). The resulting Hamiltonian H' becomes

$$H' = -\frac{1}{2m}\frac{1}{2S}\left[2\sum_{k=1}^{N}(\boldsymbol{\sigma}^{(k)} \cdot \nabla)^2\right] + m + e\phi. \tag{58}$$

By writing equation (58) explicitly, we find

$$H' = -\frac{1}{2m}(\nabla - \mathrm{i}e\mathbf{A})^2 - g_S\frac{1}{2m}(\mathbf{H} \cdot \mathbf{S}) + m + e\phi, \tag{59}$$

with the spin operator

$$\mathbf{S} \equiv \sum_{k=1}^{N}\boldsymbol{\sigma}^{(k)},$$

and the magnetic field

$$\mathbf{H} \equiv \nabla \times \mathbf{A}.$$

For a spin-S particle, the gyromagnetic ratio is

$$g_S = 1/S.$$

Next, we consider the BW equation (49) with n odd:

$$(\gamma^{(k)} \cdot \partial + \kappa_m)_\alpha{}^{\alpha_{2k-1}} \Psi_{\alpha_1 \cdots \alpha_{2k-1} \cdots \alpha_n}(x) = 0,$$

where $k = 1, \ldots, N$. The same procedure as in the case of n even is applied to the situation $n = 2S = 2N + 1$. The Schrödinger equation becomes

$$\mathrm{i}\partial_t \Psi_{u_1 \cdots u \cdots u_n}(x) = (H)_u{}^{u'} \Psi_{u_1 \cdots u' \cdots u_n}(x),$$

with

$$H = -\frac{1}{2m}\frac{1}{2S}\left[2\sum_{k=1}^{N-1}(\boldsymbol{\sigma}^{(k)} \cdot \nabla)^2 + 3(\boldsymbol{\sigma}^{(N)} \cdot \nabla)^2\right] + m.$$

The resulting Hamiltonian H' has the same form as equation (59), except with the spin operator replaced by

$$\mathbf{S} \equiv \sum_{k=1}^{N-1} \boldsymbol{\sigma}^{(k)} + \frac{3}{2}\boldsymbol{\sigma}^{(N)}.$$

We thereby obtain the gyromagnetic ratio $g_S = 1/S$.

As pointed out by Lévy-Leblond in [7], the result $g_S = 1/S$ may be seen as a consequence of the Galilean relativity, and not necessarily of the Einstein relativity.

7. Discussion

Compared with the situation in RQM, the symmetry properties of NQM have not been fully explored theoretically, and their experimental consequences, not fully utilized practically. This is partly due to the fact that NQM is not usually constrained by covariance as tightly as RQM.

We tacitly have not referred to the spins of the relevant fields in sections 2 and 3. From the Lagrangian written in terms of multi-spinor fields, we have derived the Proca equation for the vector field and the Rarita–Schwinger equation for the vector-spinor field in five-dimensional Galilean space. In order to obtain the correct number of degrees of freedom to describe a spin system, we must impose by hand an additional condition on the field. For the vector field, for example, we impose the additional condition:

$$\varphi^4(x) = 0.$$

Then the divergence-free condition tells us that

$$\varphi^5(x) = -\mathrm{i}\frac{1}{m}\sum_{k=1}^{3} \nabla_k \varphi^k(x),$$

where we have utilized the ansatz in equation (4). We thus obtain the correct number of degrees of freedom. The argument used here can be applied to the vector-spinor field and the correct number of degrees of freedom describing a spin-3/2 field is obtained.

We plan to return to the spin-statistics connection in non-relativistic quantum field theory [11]. Some papers devoted to this question have appeared recently, both in non-relativistic quantum mechanics [12] and in non-relativistic quantum field theory [13].

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Appendix A. The Galilean gamma matrices and Fierz identities

The Galilean gamma matrices satisfy the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu},$$

which reads

$$(\gamma_\mu)_\alpha{}^\gamma(\gamma_\nu)_\gamma{}^\beta + (\gamma_\nu)_\alpha{}^\gamma(\gamma_\mu)_\gamma{}^\beta = 2\eta_{\mu\nu}\delta_\alpha{}^\beta.$$

The 16 linearly independent elements are

$$\gamma_A = 1, \gamma_\mu, \sigma_{\mu\nu},$$

with

$$\sigma_{\mu\nu} = \frac{1}{2\mathrm{i}}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) = \frac{1}{\mathrm{i}}(\gamma_\mu\gamma_\nu - \eta_{\mu\nu}).$$

We can choose the Hermitian conjugates of the Galilean gamma matrices such that

$$(\gamma^\mu)^\dagger = -\xi\gamma^\mu\xi = \gamma_\mu.$$

The charge conjugation matrix is denoted by C, and its inverse, C^{-1}, satisfies

$$(C)_{\alpha\gamma}(C^{-1})^{\gamma\beta} = \delta_\alpha{}^\beta, \qquad (C^{-1})^{\alpha\gamma}(C)_{\gamma\beta} = \delta^\alpha{}_\beta,$$

It is well known that C is skew symmetric and $\gamma_\mu C$ and $\sigma_{\mu\nu}C$ are symmetric, namely,

$$(\gamma_A C)_{\alpha\beta} = \epsilon_A(\gamma_A C)_{\beta\alpha}, \tag{A.1}$$

or

$$(\gamma_A)_\beta{}^\alpha = -\epsilon_A(C^{-1}\gamma_A C)_\beta{}^\alpha,$$

where

$$\epsilon_A = \begin{cases} -1, & \text{for} \quad C, \\ +1, & \text{for} \quad \gamma_\mu C \quad \text{and} \quad \sigma_{\mu\nu}C. \end{cases}$$

The prime Fierz identity is

$$I_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = \delta_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2} = \frac{1}{4}\sum_{A=1}^{16}(\gamma_A C)_{\alpha_1\alpha_2}(C^{-1}\gamma_A)^{\beta_2\beta_1}. \tag{A.2}$$

In order to prove equation (A.2), we expand $\delta_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2}$ in terms of $(\gamma_A C)_{\alpha_1\alpha_2}$ and $(C^{-1}\gamma_B)^{\beta_2\beta_1}$:

$$\delta_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2} = \sum_{A,B} C_A{}^B(\gamma^A C)_{\alpha_1\alpha_2}(C^{-1}\gamma_B)^{\beta_2\beta_1}.$$

The coefficient $C_A{}^B$ is determined by the relationship

$$(\gamma^A C)_{\alpha\beta}(C^{-1}\gamma_B)^{\beta\alpha} = \mathrm{Tr}(\gamma^A\gamma_B) = 4\delta^A{}_B, \tag{A.3}$$

namely,

$$C_A{}^B = \tfrac{1}{16}(C^{-1}\gamma_A)^{\alpha_2\alpha_1}(\gamma^B C)_{\beta_1\beta_2}\delta_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2} = \tfrac{1}{4}\delta_A{}^B.$$

Hence,

$$\delta_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2} = \sum_{A,B}\frac{1}{4}\delta_A{}^B(\gamma^A C)_{\alpha_1\alpha_2}(C^{-1}\gamma_B)^{\beta_2\beta_1},$$
$$= \frac{1}{4}\sum_A(\gamma^A C)_{\alpha_1\alpha_2}(C^{-1}\gamma_A)^{\beta_2\beta_1}.$$

In order to prove equation (A.3), we have utilized the properties of the trace for the Galilean gamma matrices:

$$\begin{aligned}
&\mathrm{Tr}(\gamma_\mu) = 0,\\
&\mathrm{Tr}(\sigma_{\mu\nu}) = 0,\\
&\mathrm{Tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu},\\
&\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\lambda) = 0,\\
&\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}),\\
&\mathrm{Tr}(\sigma_{\mu\nu}\sigma_{\rho\sigma}) = 4(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}).
\end{aligned}$$

Note that if all γ_μ's are Hermitian, then the trace of the product of an odd number of gamma matrices is identically equal to zero. With our choice, γ_μ for $\mu = 1, 2, 3$ are Hermitian, but γ_4 and γ_5 are not. We thus need a proof for each case containing odd γ_μ's.

The prime Fierz identity, equation (A.2), yields

$$A_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = \tfrac{1}{2}\big(\delta_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2} - \delta_{\alpha_1}{}^{\beta_2}\delta_{\alpha_2}^{\beta_1}\big) = \tfrac{1}{4}(C)_{\alpha_1\alpha_2}(C^{-1})^{\beta_2\beta_1}, \tag{A.4}$$

and

$$\begin{aligned}
S_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} &= \tfrac{1}{2}\big(\delta_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2} + \delta_{\alpha_1}{}^{\beta_2}\delta_{\alpha_2}^{\beta_1}\big),\\
&= \tfrac{1}{4}\big[(\gamma^\mu C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\mu)^{\beta_2\beta_1} + \tfrac{1}{2}(\sigma^{\mu\nu}C)_{\alpha_1\alpha_2}(C^{-1}\sigma_{\mu\nu})^{\beta_2\beta_1}\big],
\end{aligned} \tag{A.5}$$

where we have utilized equation (A.1). From equations (A.4) and (A.5), we obtain further Fierz identities:

$$\begin{aligned}
[\gamma^A, \gamma^B]_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} &\equiv \tfrac{1}{4}\big[(\gamma^A)_{\alpha_1}{}^{\beta_1}(\gamma^B)_{\alpha_2}{}^{\beta_2} - (\gamma^A)_{\alpha_1}{}^{\beta_2}(\gamma^B)_{\alpha_2}{}^{\beta_1}\\
&\quad - (\gamma^B)_{\alpha_1}{}^{\beta_2}(\gamma^A)_{\alpha_2}{}^{\beta_1} + (\gamma^B)_{\alpha_1}{}^{\beta_1}(\gamma^A)_{\alpha_2}{}^{\beta_2}\big],\\
&= -\tfrac{1}{8}\big[\epsilon_B(\gamma^A\gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\gamma^B\gamma^A C)_{\alpha_1\alpha_2}\big](C^{-1})^{\beta_2\beta_1},
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
(\gamma^A, \gamma^B)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} &\equiv \tfrac{1}{4}\big[(\gamma^A)_{\alpha_1}{}^{\beta_1}(\gamma^B)_{\alpha_2}{}^{\beta_2} + (\gamma^A)_{\alpha_1}{}^{\beta_2}(\gamma^B)_{\alpha_2}{}^{\beta_1}\\
&\quad + (\gamma^B)_{\alpha_1}{}^{\beta_2}(\gamma^A)_{\alpha_2}{}^{\beta_1} + (\gamma^B)_{\alpha_1}{}^{\beta_1}(\gamma^A)_{\alpha_2}{}^{\beta_2}\big],\\
&= -\tfrac{1}{8}\big[\epsilon_B(\gamma^A\gamma^\mu\gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\gamma^B\gamma^\mu\gamma^A C)_{\alpha_1\alpha_2}\big](C^{-1}\gamma_\mu)^{\beta_2\beta_1}\\
&\quad - \tfrac{1}{16}\big[\epsilon_B(\gamma^A\sigma^{\mu\nu}\gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\gamma^B\sigma^{\mu\nu}\gamma^A C)_{\alpha_1\alpha_2}\big](C^{-1}\sigma_{\mu\nu})^{\beta_2\beta_1}.
\end{aligned} \tag{A.7}$$

Note that equations (A.4) and (A.5) can be rewritten in the forms

$$\begin{aligned}
A_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} &= [1, 1]_{\alpha_1\alpha_2}{}^{\beta_1\beta_2},\\
S_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} &= (1, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}.
\end{aligned}$$

In order to write equations (A.6) and (A.7) explicitly, we recall the following relations:

$$\begin{aligned}
&[\gamma_\mu, \gamma_\nu] = 2\mathrm{i}\sigma_{\mu\nu},\\
&[\gamma_\rho, \sigma_{\mu\nu}] = 2\mathrm{i}(\eta_{\rho\nu}\gamma_\mu - \eta_{\rho\mu}\gamma_\nu),\\
&[\sigma_{\rho\sigma}, \sigma_{\mu\nu}] = 2\mathrm{i}(\eta_{\rho\nu}\sigma_{\mu\sigma} - \eta_{\rho\mu}\sigma_{\nu\sigma} - \eta_{\sigma\nu}\sigma_{\mu\rho} - \eta_{\sigma\mu}\sigma_{\nu\rho}),
\end{aligned}$$

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu},$$
$$\{\gamma_\rho, \sigma_{\mu\nu}\} = \mathrm{i}(\gamma_\rho\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu\gamma_\rho),$$
$$\{\sigma_{\rho\sigma}, \sigma_{\mu\nu}\} = (\gamma_\rho\gamma_\sigma\gamma_\nu\gamma_\mu + \gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\rho) - 2\eta_{\rho\sigma}\eta_{\mu\nu}.$$

Below is the list of Fierz identities derived from equations (A.6) and (A.7):

$$\begin{aligned}
&[\gamma^\mu, 1]_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = 0,\\
&[\sigma^{\mu\nu}, 1]_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = 0,\\
&[\gamma^\mu, \gamma^\nu]_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = -\eta^{\mu\nu} A_{\alpha_1\alpha_2}{}^{\beta_1\beta_2},\\
&[\sigma^{\mu\nu}, \gamma^\rho]_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = -\frac{\mathrm{i}}{8}[(\gamma^\rho\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu\gamma^\rho)C]_{\alpha_1\alpha_2}(C^{-1})^{\beta_2\beta_1},\\
&[\sigma^{\mu\nu}, \sigma^{\rho\sigma}]_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = -\frac{1}{8}[(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho + \gamma^\rho\gamma^\sigma\gamma^\nu\gamma^\mu - 2\eta^{\mu\nu}\eta^{\rho\sigma})C]_{\alpha_1\alpha_2}(C^{-1})^{\beta_1\beta_2},\\
&(\gamma^\mu, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} \equiv (\Gamma^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\\
&\qquad = \frac{\mathrm{i}}{4}\big[(\sigma^{\mu\lambda}C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\lambda)^{\beta_2\beta_1} - (\gamma_\lambda C)_{\alpha_1\alpha_2}(C^{-1}\sigma^{\mu\lambda})^{\beta_2\beta_1}\big],
\end{aligned}\tag{A.8}$$

$$\begin{aligned}
(\sigma^{\mu\nu}, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} &= -\frac{\mathrm{i}}{4}\big[(\gamma^\mu C)_{\alpha_1\alpha_2}(C^{-1}\gamma^\nu)^{\beta_2\beta_1} - (\gamma^\nu C)_{\alpha_1\alpha_2}(C^{-1}\gamma^\mu)^{\beta_2\beta_1}\\
&\quad + (\sigma^{\mu\lambda}C)_{\alpha_1\alpha_2}\big(C^{-1}\sigma^\nu{}_\lambda\big)^{\beta_2\beta_1} - (\sigma^{\nu\lambda}C)_{\alpha_1\alpha_2}\big(C^{-1}\sigma^\mu{}_\lambda\big)^{\beta_2\beta_1}\big],\\
(\gamma^\mu, \gamma^\nu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} &= \eta^{\mu\nu}\frac{1}{4}\left[(\gamma^\lambda C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\lambda)^{\beta_2\beta_1} - \frac{1}{2}(\sigma^{\kappa\lambda}C)_{\alpha_1\alpha_2}(C^{-1}\sigma_{\kappa\lambda})^{\beta_2\beta_1}\right]\\
&\quad - \frac{1}{4}\big[(\gamma^\mu C)_{\alpha_1\alpha_2}(C^{-1}\gamma^\nu)^{\beta_2\beta_1} + (\gamma^\nu C)_{\alpha_1\alpha_2}(C^{-1}\gamma^\mu)^{\beta_2\beta_1}\\
&\quad - (\sigma^{\mu\lambda}C)_{\alpha_1\alpha_2}\big(C^{-1}\sigma^\nu{}_\lambda\big)^{\beta_2\beta_1} - (\sigma^{\nu\lambda}C)_{\alpha_1\alpha_2}(C^{-1}\sigma^\mu{}_\lambda)^{\beta_2\beta_1}\big],\\
(\sigma^{\mu\nu}, \gamma^\rho)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} &= \frac{1}{4}\eta^{\nu\rho}\big[(\sigma^{\mu\lambda}C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\lambda)^{\beta_2\beta_1} + (\gamma_\lambda C)_{\alpha_1\alpha_2}(C^{-1}\sigma^{\mu\lambda})^{\beta_2\beta_1}\big]\\
&\quad - \frac{1}{4}\eta^{\mu\rho}\big[(\sigma^{\nu\lambda}C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\lambda)^{\beta_2\beta_1} + (\gamma_\lambda C)_{\alpha_1\alpha_2}(C^{-1}\sigma^{\nu\lambda})^{\beta_2\beta_1}\big]\\
&\quad - \frac{1}{4}\big[(\sigma^{\mu\rho}C)_{\alpha_1\alpha_2}(C^{-1}\gamma^\nu)^{\beta_2\beta_1} - (\sigma^{\nu\rho}C)_{\alpha_1\alpha_2}(C^{-1}\gamma^\mu)^{\beta_2\beta_1}\\
&\quad + (\sigma^{\mu\nu}C)_{\alpha_1\alpha_2}(C^{-1}\gamma^\rho)^{\beta_2\beta_1} + 2(\gamma^\rho C)_{\alpha_1\alpha_2}(C^{-1}\sigma^{\mu\nu})^{\beta_2\beta_1}\big]\\
&\quad + \frac{\mathrm{i}}{16}[(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\nu\gamma^\mu)\sigma^{\kappa\lambda}C]_{\alpha_1\alpha_2}(C^{-1}\sigma_{\kappa\lambda})^{\beta_2\beta_1}\\
&\quad + \frac{\mathrm{i}}{4}\big[(\sigma^{\mu\rho}\gamma_\lambda C)_{\alpha_1\alpha_2}(C^{-1}\sigma^{\nu\lambda})^{\beta_2\beta_1} - (\sigma^{\nu\rho}\gamma_\lambda C)_{\alpha_1\alpha_2}(C^{-1}\sigma^{\mu\lambda})^{\beta_2\beta_1}\\
&\quad - (\sigma^{\mu\nu}\gamma_\lambda C)_{\alpha_1\alpha_2}(C^{-1}\sigma^{\rho\lambda})^{\beta_2\beta_1}\big],\\
(\sigma^{\mu\nu}, \sigma^{\rho\sigma})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} &= -\frac{1}{8}[(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho + \gamma^\rho\gamma^\sigma\gamma^\nu\gamma^\mu - 2\eta^{\mu\nu}\eta^{\rho\sigma})\gamma^\lambda C]_{\alpha_1\alpha_2}(C^{-1}\gamma_\lambda)^{\beta_2\beta_1}\\
&\quad - \frac{1}{16}[(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho + \gamma^\rho\gamma^\sigma\gamma^\nu\gamma^\mu - 2\eta^{\mu\nu}\eta^{\rho\sigma})\sigma^{\kappa\lambda}C]_{\alpha_1\alpha_2}(C^{-1}\sigma_{\kappa\lambda})^{\beta_2\beta_1}\\
&\quad - \frac{\mathrm{i}}{4}\big[(\sigma^{\mu\nu}\gamma^\rho C)_{\alpha_1\alpha_2}(C^{-1}\gamma^\sigma)^{\beta_2\beta_1} - (\sigma^{\mu\nu}\gamma^\sigma C)_{\alpha_1\alpha_2}(C^{-1}\gamma^\rho)^{\beta_2\beta_1}\\
&\quad + (\sigma^{\rho\sigma}\gamma^\mu C)_{\alpha_1\alpha_2}(C^{-1}\gamma^\nu)^{\beta_2\beta_1} - (\sigma^{\rho\sigma}\gamma^\nu C)_{\alpha_1\alpha_2}(C^{-1}\gamma^\mu)^{\beta_2\beta_1}\big]\\
&\quad - \frac{\mathrm{i}}{4}\big[(\sigma^{\mu\nu}\sigma^{\rho\lambda}C)_{\alpha_1\alpha_2}\big(C^{-1}\sigma^\sigma{}_\lambda\big)^{\beta_2\beta_1} - (\sigma^{\mu\nu}\sigma^{\sigma\lambda}C)_{\alpha_1\alpha_2}\big(C^{-1}\sigma^\rho{}_\lambda\big)^{\beta_2\beta_1}\\
&\quad + (\sigma^{\rho\sigma}\sigma^{\mu\lambda}C)_{\alpha_1\alpha_2}\big(C^{-1}\sigma^\nu{}_\lambda\big)^{\beta_2\beta_1} - (\sigma^{\rho\sigma}\sigma^{\nu\lambda}C)_{\alpha_1\alpha_2}\big(C^{-1}\sigma^\mu{}_\lambda\big)^{\beta_2\beta_1}\big].
\end{aligned}$$

Moreover, by using the metric tensor, we have the Fierz identities in contracted form:

$$
\begin{aligned}
&[\gamma^\lambda, \gamma_\lambda]_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = -5A_{\alpha_1\alpha_2}{}^{\beta_1\beta_2},\\
&[\sigma^{\mu\lambda}, \gamma_\lambda]_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = 0,\\
&\left[\sigma^{\mu\lambda}, \sigma^\nu{}_\lambda\right]_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = -4\eta^{\mu\nu}A_{\alpha_1\alpha_2}{}^{\beta_1\beta_2},\\
&(\gamma^\lambda, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = \tfrac{1}{4}\left[3(\gamma^\lambda C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\lambda)^{\beta_2\beta_1} - \tfrac{1}{2}(\sigma^{\kappa\lambda}C)_{\alpha_1\alpha_2}(C^{-1}\sigma_{\kappa\lambda})^{\beta_2\beta_1}\right],\\
&(\sigma^{\mu\lambda}, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} \equiv 2(\overline{\Gamma}^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\\
&\qquad = \tfrac{1}{2}\left[(\sigma^{\mu\lambda}C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\lambda)^{\beta_2\beta_1} + (\gamma_\lambda C)_{\alpha_1\alpha_2}(C^{-1}\sigma^{\mu\lambda})^{\beta_2\beta_1}\right],\\
&\left(\sigma^{\mu\lambda}, \sigma^\nu{}_\lambda\right)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = -\tfrac{1}{2}\eta^{\mu\nu}(\gamma^\lambda C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\lambda)^{\beta_2\beta_1}\\
&\qquad + \tfrac{3}{4}\left[(\gamma^\mu C)_{\alpha_1\alpha_2}(C^{-1}\gamma^\nu)^{\beta_2\beta_1} + (\gamma^\nu C)_{\alpha_1\alpha_2}(C^{-1}\gamma^\mu)^{\beta_2\beta_1}\right]\\
&\qquad + \tfrac{1}{4}\left[(\sigma^{\mu\lambda}C)_{\alpha_1\alpha_2}\left(C^{-1}\sigma^\nu{}_\lambda\right)^{\beta_2\beta_1} + (\sigma^{\nu\lambda}C)_{\alpha_1\alpha_2}\left(C^{-1}\sigma^\mu{}_\lambda\right)^{\beta_2\beta_1}\right],\\
&(\sigma^{\kappa\lambda}, \sigma_{\kappa\lambda})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = -(\gamma^\lambda C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\lambda)^{\beta_2\beta_1} + \tfrac{1}{2}(\sigma^{\kappa\lambda}C)_{\alpha_1\alpha_2}(C^{-1}\sigma_{\kappa\lambda})^{\beta_2\beta_1}.
\end{aligned}
\tag{A.9}
$$

Finally, let us summarize the Fierz identities. We will do so by showing how they can be used in order to eliminate the charge conjugation matrix C from the operators $\Lambda(\partial)$ and $d(\partial)$.

A.1. Scalars

$$(\gamma^\lambda C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\lambda)^{\beta_2\beta_1} = (1, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\gamma^\lambda, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}, \tag{A.10}$$

$$\tfrac{1}{2}(\sigma^{\kappa\lambda}C)_{\alpha_1\alpha_2}(C^{-1}\sigma_{\kappa\lambda})^{\beta_2\beta_1} = 3(1, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} - (\gamma^\lambda, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}. \tag{A.11}$$

We find a useful relation by substituting equations (A.10) and (A.11) into equation (A.9):

$$(1, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = (\gamma^\lambda, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + \tfrac{1}{2}(\sigma^{\kappa\lambda}, \sigma_{\kappa\lambda})^{\beta_1\beta_2}_{\alpha_1\alpha_2}.$$

A.2. Vectors

$$
\begin{aligned}
\tfrac{1}{2}(\sigma^{\mu\lambda}C)_{\alpha_1\alpha_2}(C^{-1}\gamma_\lambda)^{\beta_2\beta_1} &= -\mathrm{i}(\Gamma^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\overline{\Gamma}^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2},\\
\tfrac{1}{2}(\gamma_\lambda C)_{\alpha_1\alpha_2}(C^{-1}\sigma^{\mu\lambda})^{\beta_2\beta_1} &= \mathrm{i}(\Gamma^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\overline{\Gamma}^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2},
\end{aligned}
$$

where

$$
\begin{aligned}
(\Gamma^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} &= (\gamma^\mu, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2},\\
(\overline{\Gamma}^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} &= \tfrac{1}{2}(\sigma^{\mu\lambda}, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}.
\end{aligned}
$$

A.3. Tensors of rank 2

$$
\begin{aligned}
&\tfrac{1}{2}(\sigma^{\mu\lambda}C)_{\alpha_1\alpha_2}\left(C^{-1}\sigma^\nu{}_\lambda\right)^{\beta_2\beta_1} = -(\Gamma^{\mu\nu})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\overline{\Gamma}^{\mu\nu})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2},\\
&\tfrac{1}{2}(\gamma^\nu C)_{\alpha_1\alpha_2}(C^{-1}\gamma^\mu)^{\beta_2\beta_1} = \tfrac{1}{2}\eta^{\mu\nu}\left[(1, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\gamma^\lambda, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right]\\
&\qquad + (\Gamma^{\mu\nu})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\overline{\Gamma}^{\mu\nu})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2},
\end{aligned}
$$

where

$$
\begin{aligned}
&(\Gamma^{\mu\nu})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = -(\Gamma^\mu)_{\alpha_1\alpha_2}{}^{\gamma_1\gamma_2}(\Gamma^\nu)_{\gamma_1\gamma_2}{}^{\beta_1\beta_2},\\
&\text{and}\quad (\overline{\Gamma}^{\mu\nu})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = -\mathrm{i}(\Gamma^\mu)_{\alpha_1\alpha_2}{}^{\gamma_1\gamma_2}(\overline{\Gamma}^\nu)_{\gamma_1\gamma_2}{}^{\beta_1\beta_2}.
\end{aligned}
$$

A.4. Tensors of rank 3

$$\tfrac{1}{2}(\gamma^\nu C)_{\alpha_1\alpha_2}(C^{-1}\sigma^{\mu\rho})^{\beta_2\beta_1} = -\eta^{\mu\nu}\big[\mathrm{i}(\Gamma^\rho)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\overline{\Gamma}^\rho)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big] + (\Gamma^{\mu\nu\rho})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\overline{\Gamma}^{\mu\nu\rho})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2},$$

$$\tfrac{1}{2}(\sigma^{\mu\rho} C)_{\alpha_1\alpha_2}(C^{-1}\gamma^{\nu})^{\beta_2\beta_1} = \eta^{\nu\rho}\big[-\mathrm{i}(\Gamma^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\overline{\Gamma}^\mu)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big] + (\overline{\Gamma}^{\mu\nu\rho})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} - (\Gamma^{\mu\nu\rho})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2},$$

where

$$(\Gamma^{\mu\nu\rho})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = -\mathrm{i}(\Gamma^{\mu\nu})_{\alpha_1\alpha_2}{}^{\gamma_1\gamma_2}(\Gamma^\rho)_{\gamma_1\gamma_2}{}^{\beta_1\beta_2},$$
$$\text{and}\quad (\overline{\Gamma}^{\mu\nu\rho})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = -\mathrm{i}(\overline{\Gamma}^{\mu\nu})_{\alpha_1\alpha_2}{}^{\gamma_1\gamma_2}(\Gamma^\rho)_{\gamma_1\gamma_2}{}^{\beta_1\beta_2}.$$

A.5. Tensors of rank 4

$$\tfrac{1}{2}(\sigma^{\mu\rho} C)_{\alpha_1\alpha_2}(C^{-1}\sigma^{\nu\sigma})^{\beta_2\beta_1} = -\eta^{\nu\rho}\big[-(\Gamma^{\mu\sigma})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\overline{\Gamma}^{\mu\sigma})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big] + (\Gamma^{\mu\nu\rho\sigma})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} - (\overline{\Gamma}^{\mu\nu\rho\sigma})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2},$$

$$\tfrac{1}{2}(\eta^{\mu\sigma}\eta^{\rho\nu} - \eta^{\mu\nu}\eta^{\rho\sigma})\big[(1,1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\gamma_\lambda,\gamma^\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big] + \eta^{\mu\sigma}\big[(\Gamma^{\rho\nu})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\overline{\Gamma}^{\rho\nu})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big]$$
$$-\eta^{\mu\nu}\big[(\Gamma^{\rho\sigma})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\overline{\Gamma}^{\rho\sigma})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big] - \eta^{\rho\sigma}\big[(\Gamma^{\mu\nu})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\overline{\Gamma}^{\mu\nu})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big]$$
$$= (\Gamma^{\mu\nu\rho\sigma})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\overline{\Gamma}^{\mu\nu\rho\sigma})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}, \tag{A.12}$$

where

$$(\Gamma^{\mu\nu\rho\sigma})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = -\mathrm{i}(\Gamma^{\mu\nu\rho})_{\alpha_1\alpha_2}{}^{\gamma_1\gamma_2}(\Gamma^\sigma)_{\gamma_1\gamma_2}{}^{\beta_1\beta_2},$$
$$\text{and}\quad (\overline{\Gamma}^{\mu\nu\rho\sigma})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = -\mathrm{i}(\overline{\Gamma}^{\mu\nu\rho})_{\alpha_1\alpha_2}{}^{\gamma_1\gamma_2}(\Gamma^\sigma)_{\gamma_1\gamma_2}{}^{\beta_1\beta_2}.$$

Note that equation (A.12) may serve as a consistency condition.

We have made repeated use of the relation

$$(\gamma^A,\gamma^B)_{\alpha_1\alpha_2}{}^{\gamma_1\gamma_2}(\gamma^C,\gamma^D)_{\gamma_1\gamma_2}{}^{\beta_1\beta_2} = \tfrac{1}{2}\big[(\gamma^A\gamma^C,\gamma^B\gamma^D)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\gamma^A\gamma^D,\gamma^B\gamma^C)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big],$$

to obtain some expressions listed in appendix B.

Appendix B. Expressions of $\Lambda(\partial)$ and $d(\partial)$ without using the charge-conjugation operator

In this appendix, we provide the explicit expressions of the operator $\Lambda(\partial)$ and its inverse operator $d(\partial)$.

B.1. Symmetric spinor field of rank 2

$$\Lambda_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}(\partial) = \sqrt{\frac{1}{2}}\big\{(\Omega,1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + \kappa_m\big[I_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (1,1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big]\big\},$$

$$d_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}(\partial) = -\frac{\sqrt{2}}{\kappa_m}\left\{\frac{1}{2}(\Omega,\Omega)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + \left(\partial^2 - \kappa_m^2\right)\left[-I_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + \frac{1}{2}(1,1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right]\right\},$$

where we have introduced the operator Ω:

$$\Omega \equiv \gamma\cdot\partial - \kappa_m.$$

B.2. Symmetric spinor field of rank 3

$$
\begin{aligned}
\Lambda_{\alpha_1\alpha_2\alpha_3}{}^{\beta_1\beta_2\beta_3}(\partial) = \Big\{ & (\Omega, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + \kappa_m\big[I_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (1, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big] + \kappa_m \frac{5}{16}\big[(1, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} \\
& + (\gamma^\lambda, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big] + \frac{1}{\kappa_m}\frac{1}{8}\big[(\Omega, \Omega)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\Omega\gamma^\lambda, \Omega\gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big] \\
& + \frac{1}{4}\big[(\Omega, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\Omega\gamma^\lambda, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big] \\
& + \frac{1}{\kappa_m}\big(\partial^2 - \kappa_m^2\big)\frac{1}{8}\big[(1, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\gamma^\lambda, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big]\Big\}\delta_{\alpha_3}{}^{\beta_3} \\
& + \frac{1}{8}\big[(1, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\gamma^\lambda, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big](\Omega)_{\alpha_3}{}^{\beta_3} \\
& + \frac{1}{16}\big\{(\Omega, \gamma^\kappa)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\Omega\gamma^\kappa, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + \mathrm{i}(\Omega\sigma^{\kappa\lambda}, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} \\
& + \kappa_m\big[2(\gamma^\kappa, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + \mathrm{i}(\sigma^{\kappa\lambda}, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big]\big\}(\gamma_\kappa)_{\alpha_3}{}^{\beta_3} \\
& + \kappa_m\frac{1}{16}\big[(\sigma^{\kappa\lambda}, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\sigma^{\kappa\lambda}\gamma^\rho, \gamma_\rho)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big](\sigma_{\kappa\lambda})_{\alpha_3}{}^{\beta_3} \\
& + \frac{1}{32}\big[(\sigma^{\kappa\lambda}, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\sigma^{\kappa\lambda}\gamma^\rho, \gamma_\rho)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big](\Omega\sigma_{\kappa\lambda})_{\alpha_3}{}^{\beta_3},
\end{aligned}
$$

$$
\begin{aligned}
d_{\alpha_1\alpha_2\alpha_3}{}^{\beta_1\beta_2\beta_3}(\partial) = \frac{1}{\kappa_m^2}\frac{1}{16}\Big\{ & 3(\Omega, \Omega)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + 2(\Omega\gamma^\lambda, \Omega\gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + \frac{3}{4}(\Omega\sigma^{\kappa\lambda}, \Omega\sigma_{\kappa\lambda})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} \\
& + \kappa_m\left[4(\Omega, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} - 2(\Omega\gamma^\lambda, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} - \frac{3}{2}(\Omega\sigma^{\kappa\lambda}, \sigma_{\kappa\lambda})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right]\Big\}(\Omega)_{\alpha_3}{}^{\beta_3} \\
& + \frac{1}{\kappa_m^2}\frac{1}{16}\Big\{2(\Omega\gamma^\kappa, \Omega)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + \kappa_m\frac{1}{2}\big[\mathrm{i}(\sigma^{\kappa\lambda}, \Omega\gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} \\
& - \mathrm{i}(\Omega\sigma^{\kappa\lambda}, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big]\Big\}(\Omega\gamma_\kappa)_{\alpha_3}{}^{\beta_3} + \frac{1}{\kappa_m^2}\frac{1}{16}\Big\{\frac{3}{4}(\Omega\sigma^{\kappa\lambda}, \Omega)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} \\
& - \frac{1}{4}(\Omega\sigma^{\kappa\lambda}\gamma^\rho, \Omega\gamma_\rho)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} - \kappa_m\frac{1}{4}\big[(\sigma^{\kappa\lambda}, \Omega)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} \\
& + (\Omega\sigma^{\kappa\lambda}, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\sigma^{\kappa\lambda}\gamma^\rho, \Omega\gamma_\rho)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\Omega\sigma^{\kappa\lambda}\gamma^\rho, \gamma_\rho)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big]\Big\}(\Omega\sigma_{\kappa\lambda})_{\alpha_3}{}^{\beta_3} \\
& + \frac{1}{\kappa_m}\big(\partial^2 - \kappa_m^2\big)\Big\{I_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} - \frac{1}{4}\big[(1, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\gamma^\lambda, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big] \\
& - \frac{1}{\kappa_m}\frac{1}{4}\big[(\Omega, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\Omega\gamma^\lambda, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big]\Big\}\delta_{\alpha_3}{}^{\beta_3} + \frac{1}{\kappa_m^2}(\partial^2 - \kappa_m^2)\frac{1}{16} \\
& \times\Big\{\frac{1}{\kappa_m^2}\frac{3}{2}\left[5(\Omega, \Omega)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} - (\Omega\gamma^\lambda, \Omega\gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + \frac{1}{2}(\Omega\sigma^{\kappa\lambda}, \Omega\sigma_{\kappa\lambda})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right] \\
& - \frac{3}{\kappa_m}\left[3(\Omega, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + (\Omega\gamma^\lambda, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} + \frac{1}{2}(\Omega\sigma^{\kappa\lambda}, \sigma_{\kappa\lambda})_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right] \\
& - \frac{1}{2}\big[15(1, 1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} - (\gamma^\lambda, \gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\big] + \frac{3}{\kappa_m^2}\big(\partial^2 - \kappa_m^2\big)
\end{aligned}
$$

$$\times\left[3(1,1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}-(\gamma^\lambda,\gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right]\Big\}(\Omega)^{\beta_3}_{\alpha_3}+\frac{1}{\kappa_m^2}\left(\partial^2-\kappa_m^2\right)$$
$$\times\frac{1}{16}\left\{\frac{1}{\kappa_m}3\left[(\gamma^\kappa,\Omega)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}+(\Omega\gamma^\kappa,1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}-\frac{\mathrm{i}}{2}(\sigma^{\kappa\lambda},\Omega\gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right.\right.$$
$$\left.\left.-\frac{\mathrm{i}}{2}(\Omega\sigma^{\kappa\lambda},\gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right]+\mathrm{i}(\sigma^{\kappa\lambda},\gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right\}(\Omega\gamma_\kappa)_{\alpha_3}{}^{\beta_3}$$
$$+\frac{1}{\kappa_m^2}\left(\partial^2-\kappa_m^2\right)\frac{1}{16}\left\{\frac{1}{\kappa_m}6(\Omega\gamma^\kappa,\Omega)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}+4\left[(\gamma^\kappa,\Omega)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right.\right.$$
$$\left.+(\Omega\gamma^\kappa,1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right]-\frac{1}{2}\left[\mathrm{i}(\sigma^{\kappa\lambda},\Omega\gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}+7\mathrm{i}(\Omega\sigma^{\kappa\lambda},\gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right]$$
$$-\frac{1}{\kappa_m^2}\left(\partial^2-\kappa_m^2\right)3\left[(\gamma^\kappa,\Omega)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}+(\Omega\gamma^\kappa,1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}-\frac{\mathrm{i}}{2}(\sigma^{\kappa\lambda},\Omega\gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right.$$
$$\left.\left.\left.-\frac{\mathrm{i}}{2}(\Omega\sigma^{\kappa\lambda},\gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}-\kappa_m\mathrm{i}(\sigma^{\kappa\lambda},\gamma_\lambda)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right]\right]\right\}(\gamma_\kappa)_{\alpha_3}{}^{\beta_3}$$
$$+\frac{1}{\kappa_m^2}\left(\partial^2-\kappa_m^2\right)\frac{1}{16}\left\{-\frac{1}{\kappa_m^2}\frac{3}{4}\left[3(\Omega\sigma^{\kappa\lambda},\Omega)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}-(\Omega\sigma^{\kappa\lambda}\gamma^\rho,\Omega\gamma_\rho)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right]\right.$$
$$+\frac{1}{\kappa_m}\frac{3}{4}\left[(\sigma^{\kappa\lambda},\Omega)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}+(\Omega\sigma^{\kappa\lambda},1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}+(\sigma^{\kappa\lambda}\gamma^\rho,\Omega\gamma_\rho)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right.$$
$$\left.+(\Omega\sigma^{\kappa\lambda}\gamma^\rho,\gamma_\rho)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right]-\frac{1}{\kappa_m^2}\left(3\partial^2-7\kappa_m^2\right)\frac{1}{4}\left[3(\sigma^{\kappa\lambda},1)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right.$$
$$\left.\left.-(\sigma^{\kappa\lambda}\gamma^\rho,\gamma_\rho)_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\right]\right\}(\Omega\sigma_{\kappa\lambda})_{\alpha_3}{}^{\beta_3}.$$

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